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TWO-DIMENSIONAL IRROTATIONAL TRANSONIC FLOWS OF A COMPRESSIBLE FLUID

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SUMMARY

The method developed in NACA TN No. 995 has been slightly modified and extended to include flows with circulation. The essential feature of the modified method is that in analytic continuation of the solution the alteration of the singularities of the incompressible solution due to the presence of the hypergeometric functions has been taken into account. It was found that for finite Mach number the only case in which the nature of the singularity of the incompressible solution can remain unchanged is for a ratio of specific heats equal to -1.

Two particular flows, one having a finite circulation and the other having zero circulation, have been studied. Both flows were derived from the incompressible flow about an elliptic cylinder of thickness ratio 0.60. The free-stream Mach number for both cases was taken to be 0.60 in order to avoid the appearance of limiting lines. The pressure distribution for the flow without circulation has been compared with that of incompressible flow over approximately the same body. The discrepancies between the exact results and those predicted by the approximate Von Kármán-Tsien and Glauert-Prandtl formulas are so wide as to show definitely that in this case the effect of geometry cannot be ignored, as is done in both approximate formulas. In general, it seems that the effect of geometry cannot be neglected and the conventional "pressure-correction" formulas are not valid, even in the subsonic region if the body is thick, especially if there is a supersonic region in the flow.

INTRODUCTION

This report is a continuation of the work reported in NACA TN No. 995. The method developed in that report has been slightly modified and extended in the present report to include flows with circulation. The general concept and method is outlined without the mathematical details in part I. The essential feature of the modified method is that in analytic continuation of the solution the alteration of the singularities of the incompressible solution due to the presence of the hypergeometric functions has been taken into account, as fully discussed

in part II. It was found that for finite Mach number the only case in which the nature of the singularity of the incompressible solution can remain unchanged is for a ratio of specific heats equal to −1. Part III, which contains a discussion of the improvement of convergence of the power series, contains no essentially new material and is included primarily for the sake of completeness. Detailed proofs are given in appendixes A to F.

Part IV contains the study of two particular flows, one having a finite circulation and the other having zero circulation. Both are derived from the incompressible flow about an elliptic cylinder of thickness ratio 0.60. The free-stream Mach number for both cases is taken to be 0.60 in order to avoid the appearance of limiting lines. The pressure distribution for the flow without circulation has been compared with that of incompressible flow over approximately the same body. The discrepancies between the exact results and those predicted by the approximate Von Kármán–Tsien and Glauert–Prandtl formulas are so wide as to show definitely that in this case the effect of geometry cannot be ignored, as is done in both approximate formulas. In general, it seems that the effect of geometry cannot be neglected, and the conventional "pressure-correction" formulas are not valid, even in the subsonic region if the body is thick, especially if there is a supersonic region in the flow. The importance of this result cannot be overemphasized, as there is a widespread tendency in engineering practice to use simple pressure-correction formulas indiscriminately.

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I – CONCEPTS AND METHODS

General Consideration of Transonic Flows

The flow of a compressible ideal fluid about an infinite cylindrical body, unlike that of an incompressible fluid, depends on, among other conditions, the speed or Mach number at infinity. If the free-stream Mach number is below a certain value, the flow pattern will be very similar to that of an incompressible fluid even though part of the flow may be supersonic. However, as soon as the limiting Mach number is reached, the situation is entirely different. The phenomenon of major importance in this new situation is the appearance in the calculations of limiting lines in the supersonic region, characterized by the fact that the fluid particles there experience an infinite pressure gradient. It can be shown that if the assumptions of isotropy and of irrotationality of the flow are not rejected, it is impossible to continue the solution beyond these singular lines (reference 1). The failure of potential flow can be attributed to the effects of viscosity and conductivity of the fluid. Although the exact relation between the limiting line and shock

wave is still not established, there is reason to believe that in most cases the theoretical appearance of a limiting line necessarily implies the existence of a shock wave. Therefore, in practice the knowledge of the conditions under which the limiting lines appear for a given body is of paramount importance.

Chaplygin Hodograph Method

To solve the problem of a potential flow of compressible fluid, the use of the hodograph method was first suggested by P. Molenbroek (reference 2) and later by S. A. Chaplygin (reference 3). The advantage of this method is that in the case of two-dimensional potential flow it leads to a linear partial differential equation instead of a quasi-linear one, such as obtained in the physical plane. The particular solutions of this partial differential equation were found to be very similar to those for an incompressible fluid except that each consists of a hypergeometric function containing the Mach number as a parameter. Since the equation is linear, a general solution can be constructed for any assigned domain by the principle of superposition. The difficulty in connection with this method is that the solution so obtained in the hodograph plane may not transform to a "good" aerodynamic body. The problem of constructing a solution in the hodograph plane which corresponds to a desired body in the physical plane has proved to be extremely difficult.

The difficulty was partly solved by constructing a solution which, in the limit of zero Mach number, reduces to a known incompressible flow. Using this idea, Chaplygin studied the subsonic motion of a gas jet and F. Ringleb (reference 4) calculated the flow around a sharp edge. Following the same principle, Tsien (reference 5) for the first time solved the problem of a subsonic flow about a closed body, by simplifying the differential equation in such a manner that the difference between the compressible and incompressible solutions appears only in a modification of the speed scale. This method was later generalized by L. Bers (references 6 and 7) to include the flow with circulation. In the general case where the ratio of specific heats γ is not equal to -1, this simple form of speed distortion does not exist; therefore the method must be extended.

Critique of a Previous Method by Tsien and Kuo

In reference 8 the flow past a closed body as well as over a wavy surface of a gas with a characteristic constant γ greater than unity was considered. The method was based on the determination of two functions: the stream function and the transformed potential introduced through a Legendre transformation. When these two functions are properly chosen, the coordinate functions $x(q,\theta)$ and $y(q,\theta)$ are given by differentiation. Consequently, with the aid of the stream function, the flow pattern can be calculated.

The difficulty of the scheme is twofold. First, whenever a stream function and a transformed potential satisfying the conditions of continuity are constructed, they are required to satisfy the further condition of "compatibility" to ensure that they actually do belong to the same flow. In the case of circulation-free flow, this condition brings no difficulty. However, for a circulatory flow there seems to be an insufficient number of arbitrary constants to meet this stringent requirement. Second, the coexistence of two indirectly connected functions, such as the stream function and the transformed potential, automatically introduces two groups of hypergeometric functions and doubles the number of sets of unknown coefficients defining the power series. All these complications result in a large amount of labor.

In order to simplify the procedure, the velocity potential will be introduced instead of the transformed potential. Inasmuch as both the stream function and the velocity potential are directly connected by simple differential equations, the determination of one leads uniquely to that of the other. The difference between the particular integrals of these functions is simply as follows. For the stream function the particular integral contains, aside from the trigonometric function, the factor $q^{\nu}F\left(a_{\nu},b_{\nu};c_{\nu};\tau\right)$, where q is the speed of the flow and $F\left(a_{\nu},b_{\nu};c_{\nu};\tau\right)$ is the hypergeometric function. On the other hand, the particular integral of the velocity potential will have only an extra factor involving the logarithmic derivative of $F\left(a_{\nu},b_{\nu};c_{\nu};\tau\right)$. Furthermore, because of the two partial differential equations connecting them, both functions will have common coefficients. Thus, only one group of hypergeometric functions and one set of unknown constants are necessary for the complete determination of the two functions.

Construction of Solution in Hodograph Plane

Before outlining the procedure adopted here, it is perhaps proper to describe the mapping of an incompressible flow. In order to simplify the argument, the body has been assumed to be symmetrical with respect to the coordinate axes and the flow at infinity to be parallel to the major axis. Such a flow, when mapped onto the hodograph plane, will give rise to two branches of Riemann surfaces, each corresponding to one-half of the physical plane. Because of symmetry, the discussion may be restricted to, say, the left-half plane D (fig. 1). Because of the fact that the maximum and minimum velocities occur on the surface of the body, the whole field of flow is mapped into the interior of the hodograph D the boundary of which corresponds uniquely to the left-half portion of the body M'SM, and infinity corresponds to a point P on the positive axis of real values. Evidently, this point is singular.

With such a domain in the hodograph plane, it is possible to construct a compressible flow from a "similar" incompressible flow. By similar flow

is meant that at the limit of zero Mach number the compressible-flow solutions will reduce uniquely to the given incompressible flow. First, since the stagnation point, that is, the origin $\underline{S}$, is a regular point, there exists a Taylor expansion in its neighborhood. As the singular point which corresponds to the infinity of the physical plane is an interior point, it must be on the circle of convergence $\underline{C}$ of the solution. Second, the solution represented by the Taylor expansion should be continued analytically for the whole domain. This can be done, of course, either formally by analytic continuation or by solving Cauchy's initial value problem. The former, however, is not practicable. As to the latter, use can be made of the known character of the singularity of the incompressible flow to determine the form of the expansion outside the circle of convergence $\underline{C}$. If the Taylor expansion is regarded as given, the "outside" solution can be uniquely defined by the conditions of continuity across the circle of convergence $\underline{C}$. This, of course, is based on the assumption that the character of the singularity is unchanged by compressibility.

It is found, however, that the assumption is not valid. Theoretically, it can be shown that, if a Taylor expansion corresponding to incompressible flow is given, then after each term of the expansion is multiplied by a proper hypergeometric function, the resulting solution will have a logarithmic singularity in addition to those it originally possessed. This means that the "distortion" due to compressibility becomes larger for larger speeds. It can also be shown that the character of the singularity is preserved for nonvanishing Mach number if and only if the ratio of the specific heats assumes the value of -1. In order to take this effect into account, the procedure is either to start with a special body to compensate it or to eliminate it by adding an extra term to the outside solution. The former procedure is difficult. By following the latter, the part of the outside solution which, in the limit, reduces to that of the incompressible flow is regarded as known. Then the Taylor expansion and the extra term added to the outside solution can be determined by the conditions of continuity. The extra term becomes zero for zero Mach number. Thus, it is seen that the Chaplygin condition is again satisfied but the procedure has been greatly extended.

The flow with a finite circulation has also been considered. In order to simplify the mathematical problem, the circulation has been assumed to be very small. Under this assumption the effect of circulation on the solution of the incompressible flow can be represented approximately by an arbitrary combination of vortices and doublets at the singularities of the hodograph. Therefore, to the original singularities there will be superposed a logarithmic one in order to define a circulation. Since the added part due to circulation is an even function, the resultant solution will be unsymmetric with respect to the major axis.

The solution for the similar compressible flow can be constructed in the same manner, with the exception that the conditions of continuity across the circle of convergence are insufficient to determine all the

arbitrary constants, in particular those characterizing the strengths of the vortices and the doublets. These are determined from the condition that at the stagnation point $dx = 0$ and $dy = 0$ and from the geometrical condition of symmetry. The former condition is satisfied in all cases irrespective of whether there is circulation, whereas the latter is required only when the circulation is finite.

Improvement of the Convergence of Power Series

The whole problem thus hinges on the method of carrying out the actual computation. It should be pointed out that inasmuch as the solutions, in the limit, reduce to harmonic functions, the convergence of the power series, especially in the neighborhood of the circle of convergence, is generally very slow. This is particularly true of the part $\sum_{0}^{\infty} q^{-\nu} F\left(a_{\nu} - \nu, b_{\nu} - \nu; 1 - \nu; \tau\right) \sin \nu\theta$ of the outside series, because it contains the hypergeometric functions which increase rapidly with ν. This situation can be eased somewhat by introducing the asymptotic expansions of the hypergeometric functions. For, after these are substituted in the solutions, the first- and sometimes even the second-order terms can be summed. As a result, the solution in each case can be broken up into two parts, one of which is of closed form and the other is a power series with improved convergence. Owing to the fact that the dominant terms give excellent approximation in the domain of validity, the value of the term given by the power series is usually of inferior order. For practical purposes when high accuracy is not desired, the amount of labor involved can be greatly reduced.

As was pointed out in reference 8, the summed part can be identified as the "speed distortion" in the subsonic region; in the supersonic region where the differential equation changes its type, it can be interpreted as a "standing wave," depending only on the two characteristic parameters. In this case both these simple solutions are known to be inaccurate, especially in the neighborhood of sonic speed. For full discussions, see reference 8.

It must be added that in the case of the derivatives of the hypergeometric functions the dominant terms cannot give as good an approximation, as can be seen by comparing $f^{(1)}(\tau)$ with $g^{(1)}(\tau)$ in table 1. For this reason, the coordinate functions which involve the hypergeometric functions as well as their derivatives have more important correction terms and hence require the use of many more terms for the actual computations.

II – CONSTRUCTION OF A SOLUTION IN HODOGRAPH PLANE

Transformed Differential Equations and Their Particular Solutions

Let u and v be the velocity components of a two-dimensional flow, parallel respectively to the x- and the y-axis of a Cartesian system. In the case of steady, irrotational, and isentropic motion of an inviscid, nonconducting, and compressible fluid, the Eulerian equations can be integrated to give the pressure p, the density ρ, or the sonic speed c in terms of the flow speed q:

$$p = p_o\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{c_o{}^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{1}$$

$$\rho = \rho_o\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{c_o{}^2}\right)^{\frac{1}{\gamma-1}} \tag{2}$$

$$\left.\begin{aligned} c^2 &= c_o{}^2 - \frac{\gamma - 1}{2}q^2 \\ q^2 &= u^2 + v^2 \end{aligned}\right\} \tag{3}$$

where p_o, ρ_o, and c_o denote respectively the values of p, ρ, and c at the stagnation point, and γ is the ratio of the specific heats of the gas. Furthermore, because of the kinematic conditions, there exist a velocity potential φ and stream function ψ defined by

$$\left.\begin{aligned} u &= \varphi_x \\ v &= \varphi_y \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \rho u &= -\rho_o\psi_y \\ \rho v &= \rho_o\psi_x \end{aligned}\right\} \tag{5}$$

where the subscripts indicate the partial derivatives.

By use of equations (4) and (5), the partial derivatives in the expressions for $d\varphi$ and $d\psi$ can be eliminated so that $d\varphi$ and $d\psi$ can be expressed linearly in terms of dx and dy or dx and dy can be expressed linearly in terms of $d\varphi$ and $d\psi$. Furthermore, if the Jacobian function $\frac{\partial(x,y)}{\partial(u,v)}$ is finite and nonvanishing, the correspondence between the physical xy- and the hodograph uv-plane is 1 to 1. Under this condition, by regarding u and v as independent variables, the relations connecting the differentials yield the following equalities:

$$x_\theta = \frac{1}{q}\left(\cos\ \theta\varphi_\theta - \frac{\rho_o}{\rho}\sin\theta\psi_\theta\right) \tag{6}$$

$$y_\theta = \frac{1}{q}\left(\sin\theta\varphi_\theta + \frac{\rho_o}{\rho}\cos\theta\psi_\theta\right) \tag{7}$$

and

$$x_q = \frac{1}{q}\left(\cos\ \theta\varphi_q - \frac{\rho_o}{\rho}\sin\theta\psi_q\right) \tag{8}$$

$$y_q = \frac{1}{q}\left(\sin\theta\varphi_q + \frac{\rho_o}{\rho}\cos\theta\psi_q\right) \tag{9}$$

where θ is the inclination of the velocity vector to the x-axis and accordingly

$$u = q\cos\theta$$

$$v = q\sin\theta$$

Finally, the condition of integrability demands that

$$q\varphi_q = -\frac{\rho_o}{\rho}\left(1 - M^2\right)\psi_\theta \tag{10}$$

$$\varphi_\theta = \frac{\rho_o}{\rho}q\psi_q \tag{11}$$

where $M = q/c$ is the local Mach number. These fundamental systems permit the complete determination of the functions φ and ψ. For by eliminating, say, $\varphi(q,\theta)$, the resultant equation for $\psi(q,\theta)$ is

$$\left(\frac{\rho_o}{\rho} q \psi_q\right)_q + \frac{\rho_o}{\rho q}(1 - M^2)\psi_{\theta\theta} = 0 \tag{12}$$

If $\psi(q,\theta)$ is known, $\varphi(q,\theta)$ is determined uniquely, aside from a constant determined by equations (10) and (11).

The particular integrals of equation (12) are of the form:

$$\psi = \psi_\nu(q)e^{i\nu\theta}$$

where ν is a positive real number and the function $\psi_\nu(q)$ satisfies

$$\frac{d}{dq}\left(\frac{q}{\rho}\frac{d\psi_\nu}{dq}\right) - \frac{\nu^2}{\rho q}(1 - M^2)\psi_\nu = 0 \tag{13}$$

If the substitutions, according to Chaplygin,

$$\psi_\nu(q) = q^\nu F_\nu(\tau)$$

and

$$\tau = \frac{1}{2\beta}\frac{q^2}{c_o^2} \quad \text{with} \quad \beta = \frac{1}{\gamma - 1} \tag{14}$$

are made, equation (13) reduces to a familiar hypergeometric equation and $F_\nu(\tau)$ becomes one of the following integrals:

$$\left.\begin{aligned} &F(a_\nu,b_\nu;c_\nu;\tau) \\ q^{-2\nu}&F(1 + a_\nu - c_\nu,\ 1 + b_\nu - c_\nu;\ 2 - c_\nu;\ \tau) \end{aligned}\right\} \tag{15}$$

when ν differs from an integer; or

$$\left.\begin{aligned} &F(a_n,b_n;c_n;\tau) \\ q^{-2n}&\left[C_n\tau^n F(a_n,b_n;c_n;\tau)\ \log_e \tau + C_n\tau^n Q_n + P_{n-1}\right] \end{aligned}\right\} \tag{16}$$

when ν is a positive integer, where

$$Q_n = \frac{\Gamma(c_n)}{\Gamma(a_n)\Gamma(b_n)} \sum_0^\infty \Psi(a_n,b_n;m) \frac{\Gamma(a_n + m)\Gamma(b_n + m)}{\Gamma(c_n + m)\Gamma(m + 1)} \tau^m \qquad \tau < 1 \tag{17}$$

$$P_{n-1} = \frac{1}{\Gamma(n)\Gamma(a_n - n)\Gamma(b_n - n)} \sum_0^{n-1} (-1)^m \frac{\Gamma(a_n - n + m)\Gamma(b_n - n + m)\Gamma(n - m)}{\Gamma(m + 1)} \tau^m \tag{18}$$

$$\Psi(a_n,b_n;m) = \psi(a_n + m) + \psi(b_n + m) - \psi(c_n + m) - \psi(m + 1) - \psi(b_n) + \psi(c_n) \tag{19}$$

$$C_n = (-1)^{n+1} \frac{\Gamma(a_n)\Gamma(b_n)}{\Gamma(n)\Gamma(c_n)\Gamma(a_n - n)\Gamma(b_n - n)} \tag{20}$$

with

$$\left.\begin{aligned} a_\nu + b_\nu &= \nu - \beta \\ a_\nu b_\nu &= -\frac{1}{2}\beta\nu(\nu + 1) \\ c_\nu &= \nu + 1 \\ b_\nu &< 0 \end{aligned}\right\} \tag{21}$$

The hypergeometric equation thus possesses two distinct families of fundamental solutions according to whether ν differs from or is equal to an integer. The second

integral defined by expression (16) was found most appropriate for these particular parameters a_n and b_n. The reason for subtracting $C_n\left[\psi(b_n) - \psi(c_n)\right]F(a_n,b_n;c_n;\tau)$ from the second integral defined in reference 8 is to neutralize the contribution of the first integral and the discontinuities carried by $\psi(b_n)$, as b_n is negative.

In the following discussions, the two fundamental solutions of equation (13) will be denoted by $q^{\nu}F_{\nu}(\tau)$ and $q^{-\nu}F_{-\nu}(\tau)$ when ν is not an integer and by $q^nF_n(\tau)$ and $q^{-n}F_{-n}(\tau)$ when ν is an integer, where $F_{-n}(\tau)$ is defined by the expression within the bracket in expression (16). The normalization has been chosen for a continuous passage of a compressible to an incompressible flow. The most important property of $F_{-n}(\tau)$ from expression (16) is that when $n = 1$

$$F_{-1}(\tau) \equiv 1 \tag{22}$$

For this parameter the first integral reduces to

$$F_1(\tau) \equiv F(1,-\beta;2;\tau) = \frac{1}{(\beta + 1)\tau}\left[1 - (1 - \tau)^{\beta+1}\right] \tag{23}$$

The particular integrals of equation (12) are then given by

$$\left.\begin{aligned} &q^{\nu}F_{\nu}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\nu\theta \\ &q^{-\nu}F_{-\nu}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\nu\theta \end{aligned}\right\} \tag{24}$$

when ν is not an integer and by

$$\left.\begin{aligned} &q^{n}F_{n}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}n\theta \\ &q^{-n}F_{-n}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}n\theta \end{aligned}\right\} \tag{25}$$

when ν is an integer n. In addition to these solutions, there are two other integrals, each of which is a function of only one variable. On assuming $\psi = \psi(q)$ or $\psi(\theta)$, equation (12) yields

$$\theta \quad \text{and} \quad \int (1 - \tau)^{\beta} \frac{d\tau}{\tau} \tag{26}$$

Since the particular integrals of $\psi(q,\theta)$ are known, those of $\varphi(q,\theta)$ are shown to be (see appendix A)

$$\left.\begin{aligned} &(1-\tau)^{-\beta}q^{\nu}F_{\nu}(\tau)\,\xi_{\nu}(\tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\nu\theta \\ &(1-\tau)^{-\beta}q^{-\nu}F_{-\nu}(\tau)\,\xi_{\nu}(\tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\nu\theta \end{aligned}\right\} \tag{27}$$

when ν is not an integer and

$$\left.\begin{aligned} &(1-\tau)^{-\beta}q^{n}F_{n}(\tau)\,\xi_{n}(\tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\theta \\ &(1-\tau)^{-\beta}q^{-n}F_{-n}(\tau)\,\xi_{-n}(\tau)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\theta \end{aligned}\right\} \tag{28}$$

when ν is an integer n. Here the functions $\xi_{\nu}(\tau)$ and $\xi_{-n}(\tau)$ for any positive value of ν are defined by

$$\left.\begin{aligned} \nu\xi_{\nu}(\tau) &= 2\tau\,\frac{d}{d\tau}\log_e \tau^{\nu/2}F_{\nu}(\tau) \\ \nu\xi_{-\nu}(\tau) &= 2\tau\,\frac{d}{d\tau}\log_e \tau^{-\nu/2}F_{-\nu}(\tau) \end{aligned}\right\} \tag{29}$$

The following expressions corresponding to expression (26) are similarly found:

$$(1-\tau)^{-\beta} - \frac{1}{2}\int(1-\tau)^{-\beta}\,\frac{d\tau}{\tau} \quad \text{and} \quad \theta \tag{30}$$

Hydrodynamic Functions of Incompressible Flows

By following the procedure adopted in reference 8, the analysis starts with the functions required in defining an irrotational incompressible flow. In the case of an incompressible fluid for which the sonic speed is infinite, the equations satisfied by the velocity potential φ and the stream function ψ become harmonic. If $W_o(z_o)$ is the complex potential, it can be shown that

$$W_o(z_o) = \varphi_o(x_o,y_o) + i\psi_o(x_o,y_o) \quad \text{with} \quad z_o = x_o + iy_o \tag{31}$$

In a simply connected domain, the functions φ_o and ψ_o are single-valued and continuous; φ_o can be many-valued only when the flow is now free from circulation Γ_o.

If w denotes the complex velocity $u - iv$, it connects with $W_o(z_o)$ by

$$w = \frac{dW_o(z_o)}{dz_o} \equiv w(z_o) \tag{32}$$

This establishes the relation between the physical and the hodograph plane. The inverse transformation

$$z_o = z_o(w) \tag{33}$$

exists, provided that $w'(z_o) \neq 0$. This function plays an important role in the present scheme of solution and will be known as the transition function. In general, as it is an inverse function, it is not single-valued, as discussed in reference 8. By introducing this relation into equation (31), the complex potential in the hodograph plane is

$$W_o(w) = \varphi_o(u,v) + i\psi_o(u,v) \tag{34}$$

In case the solution of equation (32) is many-valued, $W_o(w)$ will represent only one branch of its many solutions. When the flow is not free from circulation, the function $w(z_o)$ will contain Γ_o as a parameter. This extra term generally makes the transition function $z_o(w)$ more complex than it would be if Γ_o vanishes. In practice, the complication can be reduced to a certain extent by a linear superposition of the two effects such that the transition function becomes

$$z_o = z_o^{(o)}(w) + \frac{\Gamma_o}{4\pi} z_o^{(1)}(w) \tag{35}$$

This simplification can be justified as long as the circulation is weak so that terms of higher order may be neglected. With the transition function so defined, the complex potential may be, similarly,

$$W_o(w) = W_o^{(o)}(w) + \frac{\Gamma_o}{4\pi} W_o^{(1)}(w) \tag{36}$$

Here $z_o^{(o)}(w)$ and $W_o^{(o)}(w)$ are respectively the transition function and complex potential for zero circulation; $z_o^{(1)}(w)$ and $W_o^{(1)}(w)$, which represent the effect of circulation, are known when $W_o(w)$ is given. It can be shown that generally $W_o^{(1)}(w)$ can be represented by vortices and doublets at the singularities of $z_o(w)$. To this order of approximation, it is easy to see that the circulation Γ_o is correctly defined.

Conversely, when $W_o(w)$ is given, $z_o(w)$ can be obtained by integration:

$$z_o(w) = \int W_o'(w) \frac{dw}{w} + \text{Constant} \tag{37}$$

which, in fact, is equivalent to equation (32).

Construction of a Symmetric Solution about the Origin

From the considerations of reference 8, if the flow about a symmetric body placed at the origin of the xy-plane is mapped onto the hodograph plane, the whole left-half plane, exterior to the body, will correspond to a region in the hodograph plane, of which the point $w = U$, U being the modulus of w at infinity of the z_o-plane, is a singularity. Then the domain within the circle $|w| = U$ is single-valued and regular. If the complex potential $W_o(w)$ is associated with a definite flow in the z_o-plane, it must be analytic and regular within $|w| = U$. Consequently, it has the Taylor expansion:

$$W_o(w) = -\sum_{0}^{\infty} A_n w^n \qquad |w| < U \tag{38}$$

where the coefficients A_n are, in general, complex. If the body is symmetrical with respect to both coordinate axes, then the coefficients are real. Separating into real and imaginary parts yields, according to equation (34),

$$\psi_o(q,\theta) = \sum_{2}^{\infty} A_n q^n \sin n\theta \tag{39}$$

$$q < 1$$

$$\varphi_o(q,\theta) = -\sum_{0}^{\infty} A_n q^n \cos n\theta \tag{40}$$

where $w = qe^{-i\theta}$ and $A_1 = 0$ because $w'(z_o) \neq 0$ at $w = 0$. From now on, z_o, w, p, and ρ are normalized in terms of a, U, p_o, and ρ_o, respectively. Then $q = 1$ at infinity, and p and ρ will be unity at $w = 0$.

According to Chaplygin's procedure, the corresponding solutions for the compressible fluid can be obtained by simply replacing the function q^n in equations (39) and (40) by $q^n F_n^{(r)}(\tau)$ and $(1 - \tau)^{-\beta} q^n F_n^{(r)}(\tau)\xi_n(\tau)$, respectively, as shown by expressions (25) and (28). The second integrals are excluded by the condition of regularity at $q = 0$. Thus the following equations are obtained:

$$\psi(q,\theta) = \sum_{2}^{\infty} A_n q^n F_n^{(r)}(\tau) \sin n\theta \tag{41}$$

$$q < 1$$

$$\varphi(q,\theta) = -(1 - \tau)^{-\beta} \sum_{2}^{\infty} A_n q^n F_n^{(r)}(\tau)\xi_n(\tau) \cos n\theta + \text{Constant} \tag{42}$$

where

$$F_n^{(r)}(\tau) \equiv \frac{F_n(\tau)}{F_n(\tau_1)} = \frac{F(a_n,b_n;c_n;\tau)}{F(a_n,b_n;c_n;\tau_1)} \tag{43}$$

and $\tau_1 = \frac{1}{2\beta}\frac{U^2}{c_o^2}$, that is, τ_1 corresponds to the free-stream velocity U. It is seen that if $c_o \rightarrow \infty$, then both τ and τ_1 tend to zero and $F_n^{(r)}(\tau) \rightarrow 1$. Thus, the solutions are reduced to the incompressible form. Furthermore, if $q \rightarrow 1$, the character of the solution is exactly like that of the incompressible solution. Hence all the specified conditions are satisfied. It must be remembered that these conditions are valid only for subsonic flows, and for this reason τ_1 is restricted to the subsonic region.

The series (equation (41)) constructed in this manner is actually convergent and represents the function $\psi(q,\theta)$ within the circle of convergence $q = 1$ (reference 8).

Analytic Continuation of Solution (Branch Point of Order 1)

In this section, it is proposed to continue the solutions ψ and φ, represented respectively by equations (41) and (42), analytically outside

the domain $|w| \leq 1$. The domain outside $|w| \leq 1$ is generally many-valued. In order to be precise, let it be a branch point of order 1. Generally, the function $W_o(w)$ has other singularities in addition to the one at $w = 1$. However, such singularities lie outside the domain of interest and thus need not be investigated. Let the nearest singularity be given by $w = V > 1$. Then, the domain to be considered outside $|w| = 1$ is an annulus with a cut joining the two singularities. The proper representation of $W_o(w)$ in such a region which has a branch point of order 1 at $w = 1$ is

$$W_o(w) = iw^{-\frac{1}{2}} W_o^*(w) \tag{44}$$

where $W_o^*(w)$ is single-valued and regular within the open annulus $1 < |w| < V$. Hence, in any closed domain $1 + \delta \leq |w| \leq V - \delta$, δ being a positive value, there exists a uniformly and absolutely convergent series:

$$W_o^*(w) = \sum_0^\infty \left(B_n^* w^n + C_n^* w^{-n}\right) \tag{45}$$

which, on substituting in equation (44), will give the continuation of the Taylor series (equation (38)). This is

$$W_o(w) = i\sum_0^\infty \left(B_n w^\nu + C_n w^{-\nu}\right) \qquad 1 < |w| < V \tag{46}$$

where the constants B_n and C_n are real because of symmetry of the body and $\nu = n + \frac{1}{2}$, n being a positive integer.

In order to continue the solution of $\psi(q,\theta)$ outside the circle $q = 1$, two alternatives are encountered. Suppose, first, that the character of the singularities of $\psi_o(q,\theta)$ defined by equation (39) is unmodified by the hypergeometric functions. Then the solution for the compressible fluid, valid in the annulus $1 < |w| < V$, can be obtained by introducing the proper hypergeometric functions corresponding to the parameter ν. The continued solution would be

$$\psi(q,\theta) = \sum_0^\infty \left[\tilde{B}_n q^\nu F_\nu^{(r)}(\tau) + \tilde{C}_n q^{-\nu} F_{-\nu}^{(r)}(\tau)\right] \cos \nu\theta \qquad 1 < q < V \tag{47}$$

where

$$\left.\begin{aligned} F_{\nu}^{(r)}(\tau) &= \frac{F_{\nu}(\tau)}{F_{\nu}(\tau_1)} \\ F_{-\nu}^{(r)}(\tau) &= \frac{F_{-\nu}(\tau)}{F_{-\nu}(\tau_1)} \end{aligned}\right\} \tag{48}$$

Here $F_{\nu}(\tau)$ and $q^{-2\nu}F_{-\nu}(\tau)$ are respectively the first and second integrals of the hypergeometric equation; $\tilde{B}_n$ and $\tilde{C}_n$ are constants. It may be added that the coefficients $\tilde{B}_n$ and $\tilde{C}_n$ are not the same as those in equation (46) for the incompressible flow but should be determined by the conditions of continuity.

Since the partial differential equation considered here is of the second order, in order to ensure that $\psi_{out}(q,\theta)$ is the analytic continuation of $\psi_{in}(q,\theta)$, two conditions must be satisfied at the boundary of the respective regions of convergence, that is, the circle $q = 1$. The two conditions are as follows:

$$\psi_{in}(q,\theta)\Big|_{q=1} = \psi_{out}(q,\theta)\Big|_{q=1} \tag{49}$$

$$\frac{\partial}{\partial q}\psi_{in}(q,\theta)\Big|_{q=1} = \frac{\partial}{\partial q}\psi_{out}(q,\theta)\Big|_{q=1} \tag{50}$$

It should be noted that the condition (equation (49)) in this particular case is identical with $\frac{\partial}{\partial \theta}\psi_{in}(q,\theta)\Big|_{q=1} = \frac{\partial}{\partial \theta}\psi_{out}(q,\theta)\Big|_{q=1}$ and that the values taken on the circle of convergence are the limiting values if such limit exists in each case. These conditions would be sufficient to determine both $\tilde{B}_n$ and $\tilde{C}_n$. However, the solution constructed by this method suffers serious distortion because, even though the values, say, ψ and ψ_o, agree on the circle $q = 1$, the tangent of ψ = Constant on $q = 1$ deviates from that of ψ_o = Constant. This, of course, will affect the character of the function $\psi_o(q,\theta)$. If $W_o(w)$ is defined by equation (38), the question arises as to the singularity after the coefficients are multiplied by the proper hypergeometric functions. The exact answer to this question has not been attempted, but a rough estimation given under theorem 1 seems sufficient for the present discussion.

<u>Theorem 1</u>.- A given Taylor expansion such as equation (38), which has a singularity at $w = 1$, is modified by multiplying its coefficients

with proper hypergeometric functions. If the hypergeometric functions can be represented in the neighborhood of $\tau = 0$ by

$$F_n(\tau) = f(\tau)T^n \left[1 + \frac{f^{(1)}(\tau)}{n} \right]$$

according to appendix C, then to its original singularities will be superposed a logarithmic one at either $w = 1$ or $w = 0$ (appendix B). This demonstrates that, in order to continue the Taylor series by the method just outlined, it is not possible to preserve the original singularity. In order to minimize this effect, the process is reversed. That is, start with a solution in $1 < q < V$ and continue the solution inwardly by satisfying the conditions of continuity on the circle $q = 1$. This is equivalent to assigning the thickness of the body first. Let the stream function $\psi(q,\theta)$ in the annulus $1 < q < V$ be of the form:

$$\psi(q,\theta) = \sum_{0}^{\infty} \left[B_n q^{\nu} F_{\nu}{}^{(r)}(\tau) + C_n q^{-\nu} F_{-\nu}{}^{(r)}(\tau) \right] \cos \nu\theta + \tilde{D}_o(\pi - \theta) + \sum_{1}^{\infty} \tilde{D}_n q^{-n} F_{-n}{}^{(r)}(\tau) \sin n\theta \quad (51)$$

where the coefficients B_n and C_n are given by equation (46) and the $\tilde{D}_n$'s are to be determined. The last two terms are introduced to counteract the distortion and are chosen so that the symmetry property of $\psi(q,\theta)$ is preserved. If $\psi(q,\theta)$ is defined by equation (41) with coefficients $\tilde{A}_n$ and equation (51) represents the solution, the constants $\tilde{A}_n$ and $\tilde{D}_n$ should satisfy, according to equations (49) and (50), on $q = 1$

$$\sum_{1}^{\infty} \left(\tilde{A}_m - \tilde{D}_m - \frac{2\tilde{D}_o}{m} \right) \sin m\theta = \sum_{0}^{\infty} \left(B_m + C_m \right) \cos \mu\theta$$

$$\sum_{1}^{\infty} m \left[\tilde{A}_m \xi_m(\tau_1) - \tilde{D}_m \xi_{-m}(\tau_1) \right] \sin m\theta = \sum_{0}^{\infty} \mu \left[B_m \xi_{\mu}(\tau_1) + C_m \xi_{-\mu}(\tau_1) \right] \cos \mu\theta$$

Here it has been tacitly assumed that the limits, as $q \to 1$, exist and $\tilde{A}_1 = 0$ in order to ensure the conditions $dx = dy = 0$ at $q = 0$, and in addition

$$\pi - \theta = 2 \sum_{1}^{\infty} \frac{\sin m\theta}{m} \qquad 0 < \theta < 2\pi$$

Multiplying both sides by $\sin n\theta$ and integrating term by term yields

$$\left.\begin{aligned} \tilde{A}_n - \tilde{D}_n - \frac{2\tilde{D}_o}{n} &= \frac{1}{\pi} \sum_{0}^{\infty} (B_m + C_m) I_{n\mu} \\ n\left[\tilde{A}_n \xi_n(\tau_1) - \tilde{D}_n \xi_{-n}(\tau_1)\right] &= \frac{1}{\pi} \sum_{0}^{\infty} \mu \left[B_m \xi_\mu(\tau_1) + C_m \xi_{-\mu}(\tau_1)\right] I_{n\mu} \end{aligned}\right\} \tag{52}$$

where

$$I_{n\mu} = \frac{1}{n + \mu} + \frac{1}{n - \mu}$$

$$\mu = m + \frac{1}{2}$$

If the right-hand members are regarded as definite quantities, the algebraic solution gives, for $n = 1$,

$$\left.\begin{aligned} \tilde{D}_o &= -\frac{1}{2\pi} \sum_{0}^{\infty} \left\{ \left[\mu \xi_\mu(\tau_1) + 1\right] B_m + \left[\mu \xi_{-\mu}(\tau_1) + 1\right] C_m \right\} I_{1\mu} \\ \tilde{D}_1 &= \frac{1}{\pi} \sum_{0}^{\infty} \mu \left[B_m \xi_\mu(\tau_1) + C_m \xi_{-\mu}(\tau_1)\right] I_{1\mu} \end{aligned}\right\} \tag{53}$$

and, for $n > 1$,

$$n\tilde{A}_n = \frac{1}{\pi}\frac{1}{\xi_n(\tau_1) - \xi_{-n}(\tau_1)}\left(\sum_0^\infty\left\{\left[\mu\xi_\mu(\tau_1) - n\xi_{-n}(\tau_1)\right]B_m\right.\right.$$

$$\left.\left.+\left[\mu\xi_{-\mu}(\tau_1) - n\xi_{-n}(\tau_1)\right]C_m\right\} - 2\pi\tilde{D}_0\xi_{-n}(\tau_1)\right) \tag{54}$$

$$n\tilde{D}_n = \frac{1}{\pi}\frac{1}{\xi_n(\tau_1) - \xi_{-n}(\tau_1)}\left(\sum_0^\infty\left\{\left[\mu\xi_\mu(\tau_1) - n\xi_n(\tau_1)\right]B_m\right.\right.$$

$$\left.\left.+\left[\mu\xi_{-\mu} - n\xi_n(\tau_1)\right]C_m\right\} - 2\pi\tilde{D}_0\xi_n(\tau_1)\right) \tag{55}$$

where the determinant $n(\xi_n - \xi_{-n}) \neq 0$. It is observed that, as $\tau_1 \to 0$, $\tilde{D}_n \to 0$ and $\tilde{A}_n \to A_n$. Consequently, Chaplygin's condition is again satisfied. Furthermore, $\tilde{D}_n = 0$ also for $\gamma = -1$. In that event, the compressible and incompressible flows will have the same singularities.

The solution is formal. In order to prove that the function $\psi(q,\theta)$ represented by equations (41) and (51) is regular in the respective domains of validity, the truth of the following theorem must first be demonstrated.

Theorem 2.- If the constants $\tilde{A}_n$ and $\tilde{D}_n$ are defined by equations (54) and (55), respectively, and if the series (equations (38) and (46)) converge uniformly and absolutely in the domains specified, the series (equation (41)) with coefficients $\tilde{A}_n$ and equation (51) are uniformly and absolutely convergent in the corresponding domains (appendix E).

With $\psi(q,\theta)$ so determined, the velocity potential, as given in the section entitled "Transformed Differential Equations and Their Particular Solutions," is

$$\varphi(q,\theta) = -(1 - \tau)^{-\beta}\sum_2^\infty \tilde{A}_n q^n F_n^{(r)}(\tau) \cos n\theta + \text{Constant} \tag{56}$$

and

$$\varphi(q,\theta) = (1-\tau)^{-\beta} \sum_{0}^{\infty} \left[B_n q^{\nu} F_{\nu}^{(r)}(\tau)\xi_{\nu}(\tau) + C_n q^{-\nu} F_{-\nu}^{(r)}(\tau)\right] \sin \nu\theta$$

$$- (1-\tau)^{-\beta} \sum_{1}^{\infty} \tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\xi_{-n}(\tau) \cos n\theta - \tilde{D}_o \left[(1-\tau)^{-\beta} - \frac{1}{2} \int_{\tau_1}^{\tau} (1-\tau)^{-\beta} \frac{d\tau}{\tau}\right] \qquad (57)$$

Thus, the velocity potential is indeterminate only to the extent of a constant, which is not essential. It must be noted, however, that the conditions of continuity on $q = 1$ can be satisfied by making use of the conditions of equations (49) and (50) together with equation (13). This shows that when the stream function $\psi(q,\theta)$ is chosen for a domain in the hodograph plane, the associated velocity potential $\varphi(q,\theta)$ can be determined uniquely.

Construction of a Solution for Flow with Circulation

The presence of circulation creates asymmetry in the flow even when the body is symmetrical. In the case of incompressible flow and as long as the circulation is weak, the asymmetry of the stream function can be characterized, according to equation (36), by the linear combination of two terms:

$$\psi_o(q,\theta) = \psi_o^{(o)}(q,\theta) + \frac{\Gamma_o}{2\pi} \psi_o^{(1)}(q,\theta) \qquad (58)$$

where $\psi_o^{(o)}(q,\theta)$ for a symmetrical body, as given by equation (39), is an odd function, whereas $\psi_o^{(1)}(q,\theta)$ is even. The complex potential $W_o^{(1)}(w)$ due to circulation must possess a logarithmic singularity in addition to poles at both $q = 1$ and $q = V$. The exact distribution of these vortices and doublets depends, of course, on the shape of the body. For convenience of discussion, it is assumed that $\psi_o^{(1)}(q,\theta)$ has the following simple representation: For $q < 1$,

$$\psi_o^{(1)}(q,\theta) = -\sum_0^\infty q^n \cos n\theta + \sum_1^\infty \frac{q^n}{n} \cos n\theta + \sum_0^\infty \left(\frac{q}{V}\right)^n \cos n\theta - \sum_1^\infty \frac{\left(\frac{q}{V}\right)^n}{n} \cos n\theta \tag{59}$$

and, for $1 < q < V$,

$$\psi_o^{(1)}(q,\theta) = \sum_1^\infty q^{-n} \cos n\theta + \sum_1^\infty \frac{q^{-n}}{n} \cos n\theta - \log_e q + \sum_0^\infty \left(\frac{q}{V}\right)^n \cos n\theta - \sum_1^\infty \frac{\left(\frac{q}{V}\right)^n}{n} \cos n\theta \tag{60}$$

Because of circulation, the function $\varphi_o^{(1)}(q,\theta)$ consists of: For $q < 1$,

$$\varphi_o^{(1)}(q,\theta) = -\sum_1^\infty q^n \sin n\theta + \sum_1^\infty \frac{q^n}{n} \sin n\theta + \sum_2^\infty \left(1 - \frac{1}{n}\right)\left(\frac{q}{V}\right)^n \sin n\theta + \tilde{A}_o^{(1)} \tag{61}$$

and, for $1 < q < V$,

$$\varphi_o^{(1)}(q,\theta) = -\sum_1^\infty q^{-n} \sin n\theta - \sum_1^\infty \frac{q^{-n}}{n} \sin n\theta - (\pi + \theta) + \sum_2^\infty \left(1 - \frac{1}{n}\right)\left(\frac{q}{V}\right)^n \sin n\theta \tag{62}$$

For a compressible flow it is assumed similarly that

$$\psi(q,\theta) = \psi^{(o)}(q,\theta) + \frac{\Gamma_o}{4\pi}\psi^{(1)}(q,\theta) \tag{63}$$

This assumption is justified only by the fact that Γ_o is small and the corresponding $\varphi^{(1)}(q,\theta)$ yields a correct circulation. In the case of a symmetrical body, $\psi^{(o)}(q,\theta)$ was

given in the two immediately preceding sections and $\psi^{(1)}(q,\theta)$ is to be determined by use of equations (59) and (60). However, in the case of the hodograph method, the solution actually is an inverse problem. A dissymmetry of the flow of a compressible fluid will contribute different distortions at the upper and lower surfaces of the body and eventually destroy its symmetry. Consequently, the contour may not even be closed. It is believed that this is the main difficulty previously experienced in connection with the hodograph method in which association of a compressible flow with incompressible flow is actually not allowed.

In the present discussion, the connection between the compressible and incompressible solutions has been abandoned and the form of the incompressible solution serves only as a guide for the construction of the compressible solution rather than as a restriction. This has been fully demonstrated in the preceding section. Accordingly, the function $\psi^{(1)}(q,\theta)$ may assume the following expression: For $q < 1$,

$$\psi^{(1)}(q,\theta) = -\sum_{0}^{\infty} A_n^{(1)} q^n F_n(\tau) \cos n\theta + \sum_{1}^{\infty} B_n^{(1)} q^n F_n(\tau) \cos n\theta$$

$$+ A_o^{(1)} \sum_{1}^{\infty} \left(1 - \frac{1}{n}\right)\left(\frac{q}{V}\right)^n F_n(r)(\tau) \cos n\theta + P(q,\theta) \tag{64}$$

and, for $1 < q < V$,

$$\psi^{(1)}(q,\theta) = \sum_{1}^{\infty} \left[C_n^{(1)} + D_n^{(1)}\right] q^{-n} F_{-n}(\tau) \cos n\theta - \frac{B_o^{(1)}}{2} \int_{\tau_1}^{\tau} (1 - \tau)^{\beta} \frac{d\tau}{\tau}$$

$$+ A_o^{(1)} \sum_{1}^{\infty} \left(1 - \frac{1}{n}\right)\left(\frac{q}{V}\right)^n F_n(\tau) \cos n\theta + P(q,\theta) \tag{65}$$

where

$$P(q,\theta) = C_o q^2 F_2(\tau) \sin 2\theta \tag{66}$$

Here the constants $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$ characterize the strength of the vortices and doublets and are determined by the conditions of continuity at $q = 1$. Generally $P(q,\theta)$ is a polynomial and is introduced to add to $\psi^{(1)}(q,\theta)$ a mixed symmetry. The number of terms required is, from cause to effect, unknown and may be different for different bodies. In the present case, only one term is taken for simplicity. On assuming that every series converges uniformly and absolutely in the respective domain of validity and tends to a definite limit as $q \to 1$, the conditions of continuity at $q = 1$ are

$$\sum_1^\infty \left[A_n^{(1)}F_n(\tau_1) + C_n^{(1)}F_{-n}(\tau_1)\right] \cos n\theta = -A_o^{(1)}$$

$$0 < \theta < 2\pi$$

$$\sum_1^\infty n\left[A_n^{(1)}F_n(\tau_1)\xi_n(\tau_1) + C_n^{(1)}F_{-n}(\tau_1)\xi_{-n}(\tau_1)\right] \cos n\theta = 0$$

and

$$\sum_1^\infty \left[B_n^{(1)}F_n(\tau_1) - D_n^{(1)}F_{-n}(\tau_1)\right] \cos n\theta = 0$$

$$0 < \theta < 2\pi$$

$$\sum_1^\infty n\left[B_n^{(1)}F_n(\tau_1)\xi_n(\tau_1) - D_n^{(1)}F_{-n}(\tau_1)\xi_{-n}(\tau_1)\right] \cos n\theta = -B_o^{(1)}(1 - \tau_1)^\beta$$

By the uniqueness theorem of expansion, since

$$\sum_1^\infty \cos n\theta = -\frac{1}{2} \qquad 0 < \theta < 2\pi$$

it follows that

$$A_n^{(1)}F_n(\tau_1) + C_n^{(1)}F_{-n}(\tau_1) = 2A_o^{(1)}$$

$$A_n^{(1)}F_n(\tau_1)\xi_n(\tau_1) + C_n^{(1)}F_{-n}(\tau_1)\xi_{-n}(\tau_1) = 0$$

and

$$B_n^{(1)}F_n(\tau_1) - D_n^{(1)}F_{-n}(\tau_1) = 0$$

$$B_n^{(1)}F_n(\tau_1)\xi_n(\tau_1) - D_n^{(1)}F_{-n}(\tau_1)\xi_{-n}(\tau_1) = \frac{2}{n}(1 - \tau_1)^{\beta}B_o^{(1)}$$

The solutions are, respectively:

$$\left.\begin{aligned} A_n^{(1)} &= -A_o^{(1)}(1 - \tau_1)^{-\beta}F_{-n}(\tau_1)\xi_{-n}(\tau_1) \\ C_n^{(1)} &= A_o^{(1)}(1 - \tau_1)^{-\beta}F_n(\tau_1)\xi_n(\tau_1) \end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned} B_n^{(1)} &= B_o^{(1)}\frac{F_{-n}(\tau_1)}{n} \\ D_n^{(1)} &= B_o^{(1)}\frac{F_n(\tau_1)}{n} \end{aligned}\right\} \tag{68}$$

From theorem 2, it can easily be shown that $\psi^{(1)}(q,\theta)$ so defined is, as assumed, uniformly and absolutely convergent in any closed domain in $q < 1$ and $1 < q < V$ and that $\psi^{(1)}(q,\theta) = 0$ when $q = 0$. Furthermore, the arbitrary constants $A_o^{(1)}$, $B_o^{(2)}$, and $C_o^{(1)}$ are to be determined by the auxiliary conditions. In the first place, the condition that $dx = dy = 0$ at $q = 0$ demands that

$$-A_1^{(1)} + B_1^{(1)} = 0$$

which leads, because of expression (22), to

$$A_o^{(1)} = (1 - \tau_1)^{\beta}B_o^{(1)} \tag{69}$$

In case there is no stagnation point, such as in the case of a body with a sharp trailing edge, this condition is again to be satisfied in order to avoid the multiplicities in x and y. The remaining two constants, namely, $B_o^{(1)}$ and $C_o^{(1)}$, will be determined by the condition of symmetry and will be discussed under Determination of Integration Constants.

Similarly, the function $\varphi(q,\theta)$ is

$$\varphi(q,\theta) = \varphi^{(o)}(q,\theta) + \frac{\Gamma_o}{4\pi}\varphi^{(1)}(q,\theta) \tag{70}$$

Here the function $\varphi^{(o)}(q,\theta)$ is given by equations (56) and (57), and $\varphi^{(1)}(q,\theta)$, for $q < 1$, is

$$\varphi^{(1)}(q,\theta) = (1 - \tau)^{-\beta}\left[\sum_{2}^{\infty} \tilde{A}_n^{(1)} q^n F_n(\tau)\xi_n(\tau) \sin n\theta + Q(q,\theta)\right] + \tilde{A}_o \tag{71}$$

and, for $1 < q < V$, is

$$\varphi^{(1)}(q,\theta) = (1 - \tau)^{-\beta}\left\{B_o^{(1)} \sum_{1}^{\infty} \left[\tilde{B}_n^{(1)} q^n F_n(\tau)\xi_n(\tau) + \tilde{C}_n^{(1)} q^{-n} F_{-n}(\tau)\xi_{-n}\right] \sin n\theta + Q\right\} - B_o^{(1)}(\pi + \theta) \tag{72}$$

where

$$\left.\begin{aligned} Q &= -C_o^{(1)} q^2 F_2(\tau)\xi_2(\tau) \cos 2\theta \\ \tilde{A}_n^{(1)} &= B_o^{(1)}\left[\xi_{-n}(\tau_1) + \frac{1}{n}\right] F_{-n}(\tau_1) + A_o^{(1)}\left(1 - \frac{1}{n}\right)\frac{1}{F_n(\tau_1)V^n} \\ \tilde{B}_n^{(1)} &= (1 - \tau_1)^{\beta}\left(1 - \frac{1}{n}\right)\frac{1}{F_n(\tau_1)V^n} \\ \tilde{C}_n^{(1)} &= \left[\xi_n(\tau_1) + \frac{1}{n}\right] F_n(\tau_1) \end{aligned}\right\} \tag{73}$$

The velocity potential thus gives, by definition, a circulation Γ:

$$\Gamma = \oint d\varphi = \Gamma_o B_o^{(1)} \tag{74}$$

where the integral is taken, in clockwise direction, about the branch point in two complete circuits.

Transition Functions

When a pair of functions $\psi(q,\theta)$ and $\varphi(q,\theta)$ are constructed in the hodograph plane, the flow pattern to which they correspond in the physical plane cannot be given directly but further integration of two differential equations is required. This process, however, does not involve any difficulty owing to the fact that both dx and dy are total differentials and consequently reduces to partial integration. In order to carry out the integration, first suppose that the functions $\psi(q,\theta)$ and $\varphi(q,\theta)$ are defined respectively by equations (41) and (56); then equations (6) and (7) become

$$\left.\begin{aligned} x_\theta &= \frac{(1-\tau)^{-\beta}}{q}\left[\sum_2^\infty n\tilde{A}_n F_n^{(r)}(\tau)\xi_n(\tau)\sin n\theta\cos\theta - \sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}\cos n\theta\sin\theta\right] \\ y_\theta &= \frac{(1-\tau)^{-\beta}}{q}\left[\sum_2^\infty n\tilde{A}_n F_n^{(r)}(\tau)\xi_n(\tau)\sin n\theta\sin\theta + \sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}\cos n\theta\cos\theta\right] \end{aligned}\right\} \quad q < 1$$

Partial integration gives, for $q < 1$,

$$\begin{aligned} x(q,\theta) = -\frac{(1-\tau)^{-\beta}}{2q}\Bigg\{&\sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}(\tau)\xi_n(\tau)\left[\frac{\cos (n-1)\theta}{n-1} + \frac{\cos (n+1)\theta}{n+1}\right] \\ &+ \sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}(\tau)\left[\frac{\cos (n-1)\theta}{n-1} - \frac{\cos (n+1)\theta}{n+1}\right]\Bigg\} + x(q) \end{aligned} \tag{75}$$

$$\begin{aligned} y(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q}\Bigg\{&\sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}(\tau)\xi_n(\tau)\left[\frac{\sin (n-1)\theta}{n-1} - \frac{\sin (n+1)\theta}{n+1}\right] \\ &+ \sum_2^\infty n\tilde{A}_n q^n F_n^{(r)}(\tau)\left[\frac{\sin (n-1)\theta}{n-1} + \frac{\sin (n+1)\theta}{n+1}\right]\Bigg\} + y(q) \end{aligned} \tag{76}$$

Differentiating equations (75) and (76) partially with respect to q and using equation (13) to eliminate the derivative higher than the first yields x_q and y_q. By comparing these two expressions with those obtained directly from equations (8) and (9) it can be concluded that

$$x'(q) = 0$$

$$y'(q) = 0$$

Therefore, both $x(q)$ and $y(q)$ are constants denoted by x_{in} and y_{in}, respectively.

On the other hand, the following expressions for $x(q,\theta)$ and $y(q,\theta)$, corresponding to $\psi(q,\theta)$ and $\varphi(q,\theta)$ defined respectively by equations (51) and (57), can be proved similarly for $1 < q < V$:

$$\begin{aligned} x(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q} \Bigg\{ & \sum_0^\infty \nu \left[B_n q^{\nu} F_\nu^{(\gamma)}(\tau) \xi_\nu(\tau) + C_n q^{-\nu} F_{-\nu}^{(\gamma)}(\tau) \xi_{-\nu}(\tau) \right] \left[\frac{\sin(\nu-1)\theta}{\nu-1} + \frac{\sin(\nu+1)\theta}{\nu+1} \right] \\ & + \sum_0^\infty \nu \left[B_n q^{\nu} F_\nu^{(\gamma)}(\tau) + C_n q^{-\nu} F_{-\nu}^{(\gamma)}(\tau) \right] \left[\frac{\sin(\nu-1)\theta}{\nu-1} - \frac{\sin(\nu+1)\theta}{\nu+1} \right] \\ & - \sum_1^\infty n\tilde{D}_n q^{-n} F_{-n}^{(\gamma)}(\tau) \xi_{-n}(\tau) \left[\frac{\cos(n-1)\theta}{n-1} + \frac{\cos(n+1)\theta}{n+1} \right] \\ & + \sum_1^\infty n\tilde{D}_n q^{-n} F_{-n}^{(\gamma)}(\tau) \left[-\frac{\cos(n-1)\theta}{n-1} + \frac{\cos(n+1)\theta}{n+1} \right] - 2\tilde{D}_0 \cos\theta \Bigg\} + x_{out} \end{aligned} \qquad (77)$$

$$
\begin{aligned}
y(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q} \Bigg\{ & \sum_{0}^{\infty} \nu \left[B_n q^{\nu} F_{\nu}^{(r)}(\tau)\xi_{\nu}(\tau) + C_n q^{-\nu} F_{-\nu}^{(r)}(\tau)\xi_{-\nu}(\tau) \right] \left[\frac{\cos(\nu-1)\theta}{\nu-1} - \frac{\cos(\nu+1)\theta}{\nu+1} \right] \\
& + \sum_{0}^{\infty} \nu \left[B_n q^{\nu} F_{\nu}^{(r)}(\tau) + C_n q^{-\nu} F_{-\nu}^{(r)}(\tau) \right] \left[\frac{\cos(\nu-1)\theta}{\nu-1} + \frac{\cos(\nu+1)\theta}{\nu+1} \right] \\
& + \sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\xi_{-n}(\tau) \left[\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1} \right] \\
& + \sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau) \left[\frac{\sin(n-1)\theta}{n-1} + \frac{\sin(n+1)\theta}{n+1} \right] - 2\tilde{D}_0 \sin\theta \Bigg\} + y_{out}
\end{aligned}
\tag{78}
$$

where x_{out} and y_{out} are constants. The terms for $n = 1$ in the third and fourth sums would have given rise to a multivalued function θ, but it is eliminated by the fact that $F_{-1}(\tau) \equiv 1$ (expression (22)). These relations establish the correspondence between the hodograph and physical planes. Inasmuch as the correspondence is 1 to 1, a line in the hodograph plane defined by

$$\psi(q,\theta) = \text{Constant} = \kappa \tag{79}$$

will correspond uniquely to a definite portion of a streamline in the physical plane. That is, eliminating one of the two variables, say, θ, from equations (75) and (76) and equations (77) and (78) by first solving equation (79), which gives

$$\theta = \theta(q,\kappa) \tag{80}$$

yields two parametric equations for the streamline $\psi = \kappa$.

In the case of circulatory flow corresponding to equations (63) and (70), the transition functions $x(q,\theta)$ and $y(q,\theta)$ can be written as

$$x(q,\theta) = x^{(o)}(q,\theta) + \frac{\Gamma_o}{4\pi} x^{(1)}(q,\theta) \tag{81}$$

$$y(q,\theta) = y^{(o)}(q,\theta) + \frac{\Gamma_o}{4\pi} y^{(1)}(q,\theta) \tag{82}$$

Here $x^{(o)}$ and $y^{(o)}$ are those quantities given from equations (75) to (78), whereas $x^{(1)}$ and $y^{(1)}$ can be obtained by the same manner from $\psi^{(1)}$ and $\varphi^{(1)}$ defined respectively by equations (64) and (71). They are, for $q < 1$,

$$\begin{aligned} x^{(1)}(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q} \Bigg\{ & \sum_{2}^{\infty} n\tilde{A}_n^{(1)} q^n F_n(\tau) \xi_n(\tau) \left[\frac{\sin(n-1)\theta}{n-1} + \frac{\sin(n+1)\theta}{n+1}\right] \\ & + \sum_{2}^{\infty} n\tilde{A}_n^{(1)} q^n F_n(\tau) \left[\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1}\right] + P_x \Bigg\} \end{aligned} \tag{83}$$

$$\begin{aligned} y^{(1)}(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q} \Bigg\{ & \sum_{2}^{\infty} n\tilde{A}_n^{(1)} q^n F_n(\tau) \xi_n(\tau) \left[\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}\right] \\ & + \sum_{2}^{\infty} n\tilde{A}_n^{(1)} q^n F_n(\tau) \left[\frac{\cos(n-1)\theta}{n-1} + \frac{\cos(n+1)\theta}{n+1}\right] + P_y \Bigg\} + y_{in}^{(1)} \end{aligned} \tag{84}$$

where

$$P_x = -2C_o^{(1)}q^2F_2(\tau)\left[\xi_2(\tau)\left(\frac{\cos 3\theta}{3} + \cos\theta\right) - \frac{\cos 3\theta}{3} + \cos\theta\right] \tag{85}$$

$$P_y = -2C_o^{(1)}q^2F_2(\tau)\left[\xi_2(\tau)\left(\frac{\sin 3\theta}{3} - \sin\theta\right) - \frac{\sin 3\theta}{3} - \sin\theta\right] \tag{86}$$

On the other hand, in the domain $1 < q < V$, they are

$$\begin{aligned} x^{(1)}(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q}\Bigg\{ & B_o^{(1)} \sum_1^\infty n\left[\tilde{B}_n^{(1)}q^nF_n(\tau)\xi_n(\tau) + \tilde{C}_n^{(1)}q^{-n}F_{-n}(\tau)\xi_{-n}(\tau)\right]\left[\frac{\sin(n-1)\theta}{n-1} \right. \\ & \left. + \frac{\sin(n+1)\theta}{n+1}\right] + B_o^{(1)} \sum_1^\infty n\left[\tilde{B}_n^{(1)}q^nF_n(\tau) + \tilde{C}_n^{(1)}q^{-n}F_{-n}(\tau)\right]\left[\frac{\sin(n-1)\theta}{n-1} \right. \\ & \left. - \frac{\sin(n+1)\theta}{n+1}\right] + P_x\Bigg\} - \frac{B_o^{(1)}}{q}\sin\theta \end{aligned} \tag{87}$$

$$\begin{aligned} y^{(1)}(q,\theta) = \frac{(1-\tau)^{-\beta}}{2q}\Bigg\{ & B_o^{(1)} \sum_1^\infty n\left[\tilde{B}_n^{(1)}q^nF_n(\tau)\xi_n(\tau) + \tilde{C}_n^{(1)}q^{-n}F_{-n}(\tau)\xi_{-n}(\tau)\right]\left[\frac{\cos(n-1)\theta}{n-1} \right. \\ & \left. - \frac{\cos(n+1)\theta}{n+1}\right] + B_o^{(1)} \sum_1^\infty n\left[\tilde{B}_n^{(1)}q^nF_n(\tau) + \tilde{C}_n^{(1)}q^{-n}F_{-n}(\tau)\right]\left[\frac{\cos(n-1)\theta}{n-1} \right. \\ & \left. + \frac{\cos(n+1)\theta}{n+1}\right] + P_y\Bigg\} + \frac{B_o^{(1)}}{q}\cos\theta + y_{out}^{(1)} \end{aligned} \tag{88}$$

Here, since $\tilde{B}_1(1) = 0$, the terms for $n = 1$ again contribute no difficulty. The constants of integration in equations (83) and (87) are left out because they can be incorporated in equations (75) and (77), respectively.

Determination of Integration Constants

Consider first the case when the flow is symmetrical. The transition functions $x(q,\theta)$ and $y(q,\theta)$ for such a flow are given respectively by equations (75) and (76) for $q < 1$ and by equations (77) and (78) for $1 < q < V$, involving four arbitrary constants. Of these four constants, x_{out} and y_{in} can be chosen arbitrarily by translating the coordinate axes. Indeed, because of symmetry, it is preferable to choose

$$x_{out}\,(q,\theta) = 0 \quad \text{for} \quad q = q_U \quad \text{and} \quad \theta = 0 \tag{89}$$

and

$$y_{in}\,(q,\theta) = 0 \quad \text{for} \quad q = 0 \tag{90}$$

whence

$$x_{out} = \frac{(1 - \tau_U)^{-\beta}}{q_U}\left\{\sum_1^{\infty} \frac{n^2}{n^2 - 1}\left[\tilde{D}_n q_U{}^{-n} F_{-n}{}^{(r)}(\tau_U)\xi_{-n}(\tau_U) + \frac{\tilde{D}_n}{n} q_U{}^{-n} F_{-n}{}^{(r)}(\tau_U)\right] + \tilde{D}_o\right\} \tag{91}$$

$$y_{in} = 0 \tag{92}$$

where q_U denotes the flow speed on the upper surface at the midsection of the body. Then x_{out} depends on τ_1 and tends to zero with τ_1.

The other two constants can now be determined by the condition of continuity at $q = 1$, namely,

$$x_{in}(1,\theta) = x_{out}(1,\theta) \tag{93}$$

$$0 < \theta < 2\pi$$

$$y_{in}(1,\theta) = y_{out}(1,\theta) \tag{94}$$

Since both limits exist, the integration of the identities yields

$$x_{in} = \frac{(1 - \tau_1)^{-\beta}}{2\pi} \left\{ \sum_0^\infty \nu \left[B_n \xi_\nu(\tau_1) + C_n \xi_{-\nu}(\tau_1) \right] \left[\frac{1}{(\nu - 1)^2} + \frac{1}{(\nu + 1)^2} \right] + \sum_0^\infty \nu (B_n + C_n) \left[\frac{1}{(\nu - 1)^2} - \frac{1}{(\nu + 1)^2} \right] \right\} + x_{out} \tag{95}$$

$$y_{out} = 0 \tag{96}$$

The fact that $y_{out} = 0$ is the consequence of $y(q,\theta) = -y(q,2\pi - \theta)$; $2x_{in}$ will be defined as the chord of the body.

For the flow with circulation, according to the preceding section, there are six constants instead of four. Let those four arising from integration be considered first. Corresponding to equations (89) and (90) the following equations may be chosen:

$$x_{out}^{(o)} = \frac{(1 - \tau_U)^{-\beta}}{q_U} \left\{ \sum_1^\infty \frac{n}{n^2 - 1} \tilde{D}_n q_U^{-n} F_{-n}(\tau_U) \left[n\xi_{-n}(\tau_U) + 1 \right] + \tilde{D}_o - \frac{\Gamma_o}{8\pi} P_x(q_U, 0) \right\} \tag{97}$$

$$y_{out}^{(1)} = 0 \tag{98}$$

Then the condition of continuity at $q = 1$ gives

$$x_{in}^{(o)} = \frac{(1 - \tau_1)^{-\beta}}{2\pi} \left\{ \sum_0^\infty \nu \left[B_n \xi_\nu(\tau_1) + C_n \xi_{-\nu}(\tau_1) \right] \left[\frac{1}{(\nu - 1)^2} + \frac{1}{(\nu + 1)^2} \right] + \sum_0^\infty \nu (B_n + C_n) \left[\frac{1}{(\nu - 1)^2} - \frac{1}{(\nu + 1)^2} \right] \right\} + x_{out}^{(o)} \tag{99}$$

$$y_{in}^{(1)} = -B_o^{(1)} + \frac{A_o^{(1)}}{F_1(\tau_1)V} \tag{100}$$

In order to derive the second term in equation (100), use has been made of the fact that $F_1(\tau) - \frac{\beta\tau}{2}F_{1,1}(\tau) = (1 - \tau)^{\beta}$, which is just the Wronskian of the particular integrals of the hypergeometric equation for $n = 1$.

The arbitrary constants $B_o^{(1)}$ and $C_o^{(1)}$, on the other hand, are determined by an entirely different consideration. The fact that $B_o^{(1)}$ and $C_o^{(1)}$ are different from unity and zero, respectively, as they would be if τ_1 tends to zero is due to the fact that the distortion produced by compressibility is nonuniform at the surfaces of the body. In order to correct this defect completely, a more elaborate method would have been required. For the present simple investigation in which symmetry is not strictly satisfied even in the limiting case of zero Mach number, the condition of symmetry will be applied to only a few selected points. First, let it be required that

$$\left.\begin{aligned} x_{out}(q_U,0) &= 0 \\ x_{out}(q_L,2\pi) &= 0 \end{aligned}\right\} \tag{101}$$

where q_L stands for the flow speed on the lower surface at the midsection of the body. These two equations determine uniquely $C_o^{(1)}$, namely,

$$\frac{\Gamma_o}{6\pi}C_o^{(1)} = -\frac{\delta(q_U) - \delta(q_L)}{e_2(q_U) - e_2(q_L)} \tag{102}$$

where

$$\delta(q) = \frac{(1 - \tau)^{-\beta}}{q}\left[\sum_{1}^{\infty} \frac{n}{n^2 - 1}\tilde{D}_n q^{-n} F_{-n}(\tau)\left(n\xi_{-n} + 1\right) + \tilde{D}_o\right] \tag{103}$$

$$e_2(q) = (1 - \tau)^{-\beta} q F_2(\tau)\left[2\xi_2(\tau) + 1\right] \tag{104}$$

The value of $C_o^{(1)}$ becomes zero with τ_1 for $\tilde{D}_n \rightarrow 0$ as $\tau_1 \rightarrow 0$, as defined in equation (55).

The maximum distortion with respect to the y-axis will occur at the midsection. If the required body is assumed to be symmetrical, that is,

$$y(q_U,0) = \left| y(q_L,2\pi) \right|$$

the circulation will be so adjusted that

$$\Gamma_o B_o^{(1)} = 4\pi \frac{y^{(o)}(q_L,0) - y^{(o)}(q_U,0)}{y^{(1)}(q_U) + y^{(1)}(q_L)} \tag{105}$$

where $y^{(o)}(q,0) > 0$ is defined by equation (78) for $\theta = 0$ and $y^{(1)}(q)$ is given by

$$y^{(1)}(q) = \frac{(1-\tau)^{-\beta}}{q} \left\{ \sum_1^\infty \frac{n}{n^2 - 1} \left[\tilde{B}_n^{(1)} q^n F_n(\tau) \xi_n(\tau) + \tilde{C}_n^{(1)} q^{-n} F_{-n}(\tau) \xi_{-n}(\tau) \right] \right.$$
$$\left. + \sum_1^\infty \frac{n^2}{n^2 - 1} \left[\tilde{B}_n^{(1)} q^n F_n(\tau) + \tilde{C}_n^{(1)} q^{-n} F_{-n}(\tau) \right] \right\} + \frac{1}{q} \tag{106}$$

Thus, the circulation in the case of compressible flow is generally connected directly with the shape of the body. If, however, the camber is not controlled, $A_o^{(1)}$ may be taken equal to unity, by analogy with the case of incompressible flow. Then the condition of conformality (equation (69)) becomes the only condition to define the circulation. The relative merit of these two procedures can be ascertained only by comparing the numerical results.

III - IMPROVEMENT OF CONVERGENCE OF SOLUTION BY ASYMPTOTIC PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

Transformation of Stream Function $\psi(q,\theta)$

The stream function $\psi(q,\theta)$ for a flow which was derived from $\psi_o(q,\theta)$ with a branch point of order 1 has been given. The form of representation is not, in general, suitable for practical calculation. The difficulty is twofold: First, the series involves an infinite number of hypergeometric functions which are, in turn, defined as infinite series. The convergence of the hypergeometric series in this particular instance decreases with increase of the parameter ν. This means that the computation for the later terms of the series for $\psi(q,\theta)$ will be increasingly laborious. Second, the convergence of the power series defining the function $\psi(q,\theta)$ is, as expected, very slow in the neighborhood of the circle of convergence. In order to render the method of any practical value, the task of transforming the series into a more rapidly convergent one is encountered. For this purpose the following procedure is adopted.

The stream function $\psi_o(q,\theta)$ for the similar incompressible flow is (see section entitled "Construction of a Symmetric Solution about the Origin")

$$\psi_o(q,\theta) = \sum_{2}^{\infty} A_n q^n \sin n\theta \qquad q < 1$$

which is absolutely and uniformly convergent in any closed domain in $q < 1$. Then it can justly be thought of as representing not merely a regular but a closed function. In doing this, of course, a large class of problems is automatically eliminated and those cases are then considered in which simple representation of both $\varphi_o(q,\theta)$ and $\psi_o(q,\theta)$ exists. This is justified only by the mounting difficulties faced in carrying out such a detailed investigation.

It is thus observed that the difference between the stream functions $\psi(q,\theta)$ and $\psi_o(q,\theta)$ lies only in the appearance, in the former, of the hypergeometric functions. If, however, approximation is allowed, then, according to expression (223), first let $F_n(\tau)$ be substituted by its asymptotic expression, namely,

$$F_n^{(r)}(\tau) \sim \frac{f(\tau)}{f(\tau_1)} t^n(\tau) \left[1 + O\left(\frac{1}{n}\right)\right] \qquad n > N \qquad (107)$$

where $t(\tau) = \frac{T(\tau)}{T(\tau_1)}$. Here the term involving $f^{(1)}(\tau) << 1$ has been neglected. Furthermore, as will be shown in equation (159), to the same order of approximation the coefficient $\widetilde{A}_n$ can be written as

$$\widetilde{A}_n \sim A_n + O\left(\frac{1}{n}\right) \qquad n > N \tag{108}$$

By substituting the approximate values in equation (41) it can be shown that

$$\psi(q,\theta) \sim \frac{f(\tau)}{f(\tau_1)} \psi_o(qt,\theta) \tag{109}$$

That is, to this order of approximation the power series representing the stream function $\psi(q,\theta)$ can be summed and is given by expression (109).

As was shown in reference 8, the asymptotic representation is valid only when the parameter n is large. Namely, during the summation of expression (109) the value neglected becomes smaller as n increases and approaches zero as n tends to infinity. This concentration of errors in the lower-order terms makes it especially easy to apply the correction if high accuracy is required. In doing this, the quantity given by expression (109) can be added and, at the same time, subtracted from $\psi(q,\theta)$. Then a simple manipulation shows that

$$\psi(q,\theta) = \psi_1(q,\theta) + \psi_2(q,\theta) \tag{110}$$

where

$$\psi_1(q,\theta) = \frac{f(\tau)}{f(\tau_1)} \psi_o(qt,\theta) \tag{111}$$

$$\psi_2(q,\theta) = \sum_2^\infty G_n(\tau)q^n \sin n\theta \qquad q < 1 \tag{112}$$

with

$$G_n(\tau) = F_n(\tau)\Delta\widetilde{A}_n + \frac{A_n}{f(\tau_1)T^n(\tau_1)}\Delta F_n(\tau) \tag{113}$$

$$\left.\begin{aligned} \Delta\widetilde{A}_n &= \frac{\widetilde{A}_n}{F_n(\tau_1)} - \frac{A_n}{f(\tau_1)T^n(\tau_1)} \\ \Delta F_n(\tau) &= F_n(\tau) - f(\tau)T^n(\tau) \end{aligned}\right\} \tag{114}$$

Here n is a positive integer. The stream function $\psi(q,\theta)$ is then represented by the sum of two functions $\psi_1(q,\theta)$ and $\psi_2(q,\theta)$. Of these, $\psi_1(q,\theta)$ is of closed form, which differs from $\psi_o(q,\theta)$ only by a change of scale of q, and $\psi_2(q,\theta)$ is a difference of two absolutely and uniformly convergent series and hence is absolutely and uniformly convergent. In fact, according to expressions (107) and (108), $G_n(\tau)$ is of order $\frac{A_n}{n}t^n(\tau)$; the convergence of $\psi(q,\theta)$ is therefore increased by $1/n$. This actually is the gist of the whole problem.

In the annulus region $1 < q < V$; on the other hand, the stream function $\psi_o(q,\theta)$ is represented by

$$\psi_o(q,\theta) = \sum_0^\infty \left(B_n q^\nu + C_n q^{-\nu}\right) \cos \nu\theta$$

and the stream function $\psi(q,\theta)$ of the compressible flow is

$$\psi(q,\theta) = \sum_0^\infty \left[B_n q^\nu F_\nu^{(r)}(\tau) + C_n q^{-\nu} F_{-\nu}^{(r)}(\tau)\right] \cos \nu\theta$$

$$+ \tilde{D}_o(\pi - \theta) + \sum_1^\infty \tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau) \sin n\theta$$

where B_n and C_n are assumed to be given by $\psi_o(q,\theta)$, and $F_\nu^{(r)}(\tau)$ stands for the ratio of $F_\nu(\tau)$ to $F_\nu(\tau_1)$. In this region, the hypergeometric functions will be of mixed character. If the critical point $\tau_c = \frac{1}{2\beta + 1}$ is not reached, they are of exponential type; beyond this point they will change over into oscillatory type. If τ_c lies in the range $1 < q < V$, the singularity of the asymptotic expansions of the hypergeometric functions will certainly be found inside the domain in question. If this neighborhood is excluded and the hypergeometric functions are first substituted by their asymptotic forms given in expressions (223) and (224), it can similarly be shown that, for $\tau_1 < \tau < \frac{1}{2\beta + 1}$,

$$\psi(q,\theta) = \psi_1(q,\theta) + \psi_2(q,\theta) + \psi_3(q,\theta) \tag{115}$$

where

$$\psi_1(q,\theta) = \frac{f(\tau)}{f(\tau_1)} \psi_o(q,\theta)$$

$$\psi_2(q,\theta) = \sum_0^\infty \left[B_n H_\nu(\tau) q^\nu + C_n H_{-\nu}(\tau) q^{-\nu}\right] \cos \nu\theta \tag{116}$$

$$\psi_3(q,\theta) = \tilde{D}_0(\pi - \theta) + \sum_1^\infty \tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau) \sin n\theta \tag{117}$$

and

$$\left.\begin{aligned} H_\nu(\tau) &= F_\nu(\tau)\Delta F_\nu^{-1}(\tau_1) + \frac{1}{f(\tau_1)T^\nu(\tau_1)}\Delta F_\nu(\tau) \\ H_{-\nu}(\tau) &= F_{-\nu}(\tau)\Delta F_{-\nu}^{-1}(\tau_1) + \frac{1}{f(\tau_1)T^{-\nu}(\tau_1)}\Delta F_{-\nu}(\tau) \end{aligned}\right\} \tag{118}$$

with

$$\Delta F_\nu^{-1}(\tau_1) = \frac{1}{F_\nu(\tau_1)} - \frac{1}{f(\tau_1)T_\nu(\tau_1)}$$

$$\Delta F_\nu(\tau) = F_\nu(\tau) - f(\tau)T^\nu(\tau)$$

In equation (115) $\psi_1(q,\theta)$ again represents a closed function $\psi_0(qt,\theta)$, and $\psi_2(q,\theta)$, an absolutely and uniformly convergent series with improved convergence. The fact that $\psi_3(q,\theta)$ is not summed is due to the fact that $\tilde{D}_n$ is of inferior order as compared with $\tilde{A}_n$. Even though $\tilde{D}_n$ decreases as $1/n$, $\psi_3(q,\theta)$ will require fewer terms than $\psi_2(q,\theta)$ although its coefficients behave like $n^{-3/2}$ (provided that the flow $\psi_0(x_0,y_0)$ is of the nature of a doublet). Here the functions $H_\nu(\tau)$ are so defined that the functions of τ_1 and those of τ are separated so as to make possible the tabulation of $\Delta F\nu(\tau)$ and $\Delta F_{-\nu}(\tau)$.

In the region $\frac{1}{2\beta + 1} < \tau < 1$, on the other hand, $F_\nu(\tau)$ and $F_{-\nu}(\tau)$ are replaced respectively by

$$\frac{1}{2} f(\tau)T^\nu(\tau) \cos \left(\nu\omega - \frac{\pi}{4}\right)$$

$$\frac{1}{2} f(\tau)T^{-\nu}(\tau) \cos \left(\nu\omega + \frac{\pi}{4}\right)$$

where $f(\tau)$, $T(\tau)$, and $\omega(\tau)$ are given in table 1. The factor 1/2 is introduced before the first expression for symmetry. By writing

$$2 \cos \nu\theta \cos \left(\nu\omega - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \left(\cos \nu\xi + \cos \nu\eta + \sin \nu\xi - \sin \nu\eta\right)$$

$$2 \cos \nu\theta \cos \left(\nu\omega + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \left(\cos \nu\xi + \cos \nu\eta - \sin \nu\xi + \sin \nu\eta\right)$$

with

$$\left.\begin{aligned} \xi &= \theta + \omega \\ \eta &= \theta - \omega \end{aligned}\right\} \tag{119}$$

the following expression, corresponding to equation (111), is obtained

$$\psi(q,\theta) \sim \frac{f(\tau)}{2^{5/2} f(\tau_1)} \left[\psi_o(\lambda,\xi) + \psi_o(\lambda,\eta) + \varphi_o(\lambda,\xi) - \varphi_o(\lambda,\eta)\right]$$

where λ is a constant defined by

$$\lambda = \frac{2(2\beta)^{\alpha/2}}{(1 + \alpha)^{\alpha}(2\beta\tau_1)^{1/2}} \frac{1}{T(\tau_1)} > 1 \tag{120}$$

as from equation (228) $qt = \lambda U$ if $\frac{1}{2\beta + 1} < \tau < 1$. The constant λ then is a function of the free-stream Mach number and the characteristic constant of the gas but independent of the shape of the body. By eliminating the error introduced during summation, the stream function $\psi(q,\theta)$ in the supersonic range $\frac{1}{2\beta + 1} < \tau < 1$ is

$$\psi(q,\theta) = \psi_1(q,\theta) + \psi_2(q,\theta) + \psi_3(q,\theta)$$

where

$$\psi_1(q,\theta) = 2^{-5/2} \frac{f(\tau)}{f(\tau_1)} \left[\psi_o(\lambda,\xi) + \psi_o(\lambda,\eta) + \varphi_o(\lambda,\xi) - \varphi_o(\lambda,\eta)\right] \tag{121}$$

$$\psi_2(q,\theta) = \sum_0^{\infty} \left[B_n H_\nu(\tau) q^\nu + C_n H_{-\nu}(\tau) q^{-\nu} \right] \cos \nu\theta \tag{122}$$

Here $\psi_3(q,\theta)$ is given by equation (117) and $H_\nu(\tau)$ and $H_{-\nu}(\tau)$, by equation (118), except that $\Delta F_\nu(\tau)$ and $\Delta F_{-\nu}(\tau)$ are defined by

$$\Delta F_{\pm\nu}(\tau) = F_{\pm\nu}(\tau) - \frac{1}{2}f(\tau)T^{\pm\nu} \cos \left(\nu\omega \mp \frac{\pi}{4}\right) \tag{123}$$

Unlike the previous calculations, $H_\nu(\tau)$ in equation (122) is not of the order of $1/\nu$, owing to the presence of 1/2 in front of $f(\tau)T^\nu \cos \left(\nu\omega - \frac{\pi}{4}\right)$. This, however, does not introduce a serious objection, as the series in which it appears now behaves like $B_n\lambda^\nu$, which, according to equation (120) $(\lambda > 1)$, converges more rapidly than B_nq^ν in $\psi_o(q,\theta)$.

In the hyperbolic domain, moreover, the function $\psi_1(q,\theta)$ depends, aside from a factor $f(\tau)$, only on the two independent families of the characteristic parameters ξ and η defined by equation (119). This result is most striking, as it shows that the main part of the solution satisfies the simple wave equation and thus clearly demonstrates its hyperbolic character. With both the incompressible stream function $\psi_o(q,\theta)$ and the incompressible velocity potential $\varphi_o(q,\theta)$ appearing in the solution, it is impossible to establish a simple relation between the incompressible streamlines and the compressible streamlines. Since such a simple relation is the foundation of the so-called speed correction formula for a quick estimation of velocity distribution in compressible flow from that of incompressible flow over the same body, this idea cannot be extended to supersonic regions. On the other hand, this also indicates that, although the differential equation for $\psi(q,\theta)$ is hyperbolic in the supersonic range, it cannot be reduced to the simple wave equation by a mere distortion of the speed scale as given by the function $\omega(\tau)$. For if this were the case, then $\psi_1(q,\theta)$ would constitute an exact solution without the additional $\psi_2(q,\theta)$. This fact is all the more important as the additional $\psi_2(q,\theta)$ is not small in comparison with $\psi_1(q,\theta)$ for the mixed subsonic and supersonic flows, especially for the transitional region near the sonic velocity. However, in the case of pure supersonic flow, $\psi_2(q,\theta)$ might be small; then $\psi_1(q,\theta)$ alone may be used as a satisfactory approximation.

Transformation of Coordinate Functions $x(q,\theta)$ and $y(q,\theta)$

From equations (75) and (76) it is seen that the coefficients of the series defining the coordinate functions $x(q,\theta)$ and $y(q,\theta)$ are of the same order of magnitude as those of the series for the stream function $\psi(q,\theta)$ (equation (41)). By analogy with the preceding section the first-order terms in both x and y can be similarly summed. First, let them be written, for $q < 1$, as

$$x(q,\theta) = -\frac{(1-\tau)^{-\beta}}{q}\left\{\sum_{2}^{\infty}\frac{n\tilde{A}_n}{n-1}q^{n}F_n^{(\gamma)}(\tau)\cos(n-1)\theta - \frac{\beta\tau}{2}\sum_{2}^{\infty}n\tilde{A}_n q^{n}F_{n,1}^{(\gamma)}(\tau)\left[\frac{\cos(n-1)\theta}{n-1} + \frac{\cos(n+1)\theta}{n+1}\right]\right\} + x_{in} \tag{124}$$

$$y(q,\theta) = \frac{(1-\tau)^{-\beta}}{q}\left\{\sum_{2}^{\infty}\frac{n\tilde{A}_n}{n-1}q^{n}F_n^{(\gamma)}(\tau)\sin(n-1)\theta - \frac{\beta\tau}{2}\sum_{2}^{\infty}n\tilde{A}_n q^{n}F_{n,1}^{(\gamma)}(\tau)\left[\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1}\right]\right\} \tag{125}$$

Now for the similar incompressible flow with complex potential defined by equation (38) the coordinate functions can easily be deduced as

$$x_o(q,\theta) = -\sum_{2}^{\infty}\frac{nA_n}{n-1}q^{n-1}\cos(n-1)\theta + x_{in}(o) \tag{126}$$

$$q < 1$$

$$y_o(q,\theta) = \sum_{2}^{\infty}\frac{nA_n}{n-1}q^{n-1}\sin(n-1)\theta \tag{127}$$

Furthermore,

$$\frac{\cos(n-1)\theta}{n-1} + \frac{\cos(n+1)\theta}{n+1} = \frac{2n}{n^2-1}\left(\cos n\theta\cos\theta + \frac{\sin n\theta\sin\theta}{n}\right) \tag{128}$$

$$n > 1$$

$$\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1} = -\frac{2n}{n^2-1}\left(\cos n\theta\sin\theta - \frac{\sin n\theta\cos\theta}{n}\right) \tag{129}$$

If the hypergeometric functions are substituted by the dominant term of their asymptotic expansions and the coefficients are approximated by A_n, then $x(q,\theta)$ and $y(q,\theta)$ reduce to

$$x(q,\theta) \sim \frac{-(1-\tau)^{-\beta}}{q}\left[\frac{f(\tau)}{f(\tau_1)} \sum_2^{\infty} \frac{nA_n}{n-1}(qt)^n \cos (n-1)\theta\right.$$

$$\left. - \beta\tau\frac{g(\tau)}{f(\tau_1)} \sum_2^{\infty} A_n(qt)^n\left(\cos n\theta \cos \theta + \frac{\sin n\theta \sin \theta}{n}\right)\right] + x_{in}$$

$$y(q,\theta) \sim \frac{(1-\tau)^{-\beta}}{q}\left[\frac{f(\tau)}{f(\tau_1)} \sum_2^{\infty} \frac{nA_n}{n-1}(qt)^n \sin (n-1)\theta\right.$$

$$\left. + \beta\tau\frac{g(\tau)}{f(\tau_1)} \sum_2^{\infty} A_n(qt)^n\left(\cos n\theta \sin \theta - \frac{\sin n\theta \cos \theta}{n}\right)\right]$$

According to the indentities (equations (126), (127), (40), and (251)), it is evident that

$$x(q,\theta) \sim (1-\tau)^{-\beta}\left(\frac{f(\tau)t(\tau)}{f(\tau_1)}\left[x_o(qt,\theta) - x_{in}(o)\right]\right.$$

$$\left. - \frac{\beta\tau}{q}\frac{g(\tau)}{f(\tau_1)}\left\{\left[\varphi_o(qt,\theta) + A_o\right] \cos \theta - X(qt,\theta) \sin \theta\right\}\right) + x_{in}$$

$$y(q,\theta) \sim (1-\tau)^{-\beta}\left(\frac{f(\tau)t(\tau)}{f(\tau_1)}\left[y_o(qt,\theta)\right] - \frac{\beta\tau}{q}\frac{g(\tau)}{f(\tau_1)}\left\{\left[\varphi_o(qt,\theta)\right.\right.\right.$$

$$\left.\left.\left. + A_o\right] \sin \theta + X(qt,\theta) \cos \theta\right\}\right)$$

In order to eliminate the error introduced, equations (124) and (125) can similarly be written as

$$x(q,\theta) = x_1(q,\theta) + x_2(q,\theta) \tag{130}$$

$$y(q,\theta) = y_1(q,\theta) + y_2(q,\theta) \tag{131}$$

Here $x_1(q,\theta)$ and $y_1(q,\theta)$ represent

$$x_1(q,\theta) = (1 - \tau)^{-\beta}\left\{\frac{f(\tau)t(\tau)}{f(\tau_1)}x_o(qt,\theta) - \frac{\beta\tau}{q}\frac{g(\tau)}{f(\tau_1)}\left[\varphi_o(qt,\theta)\cos\theta - \chi(qt,\theta)\sin\theta\right]\right\} \tag{132}$$

$$y_1(q,\theta) = (1 - \tau)^{-\beta}\left\{\frac{f(\tau)t(\tau)}{f(\tau_1)}y_o(qt,\theta) - \frac{\beta\tau}{q}\frac{g(\tau)}{f(\tau_1)}\left[\varphi_o(qt,\theta)\sin\theta + \chi(qt,\theta)\cos\theta\right]\right\} \tag{133}$$

and $x_2(q,\theta)$ and $y_2(q,\theta)$ are series valid for $q < 1$:

$$\begin{aligned} x_2(q,\theta) = &-\frac{(1 - \tau)^{-\beta}}{q}\left[\sum_2^\infty \frac{n}{n - 1}G_n(\tau)q^n\cos(n - 1)\theta - \beta\tau\sum_2^\infty G_{n,1}(\tau)q^n\left(\cos n\theta\cos\theta\right.\right.\\ &\left.\left.+\frac{\sin n\theta\sin\theta}{n}\right) + A_o\beta\tau\frac{g(\tau)}{f(\tau_1)}\cos\theta\right] + \Delta x_{in} \end{aligned} \tag{134}$$

$$\begin{aligned} y_2(q,\theta) = &\frac{(1 - \tau)^{-\beta}}{q}\left[\sum_2^\infty \frac{n}{n - 1}G_n(\tau)q^n\sin(n - 1)\theta + \beta\tau\sum_2^\infty G_{n,1}(\tau)q^n\left(\cos n\theta\sin\theta\right.\right.\\ &\left.\left.-\frac{\sin n\theta\cos\theta}{n}\right) - A_o\beta\tau\frac{g(\tau)}{f(\tau_1)}\sin\theta\right] \end{aligned} \tag{135}$$

where $G_n(\tau)$ is defined by equation (113), and $G_{n,1}(\tau)$ is given by

$$G_{n,1}(\tau) = F_{n,1}(\tau)\,\Delta\frac{n^2\tilde{A}_n}{(n^2-1)F_n(\tau_1)} + \frac{A_n}{f(\tau_1)T^n(\tau_1)}\,\Delta F_{n,1}(\tau) \tag{136}$$

with

$$\left.\begin{aligned}
\Delta F_{n,1}(\tau) &= F_{n,1}(\tau) - g(\tau)T^n(\tau)\\
\Delta\frac{n^2\tilde{A}_n}{(n^2-1)F_n(\tau_1)} &= \frac{n^2\tilde{A}_n}{(n^2-1)F_n(\tau_1)} - \frac{A_n}{f(\tau_1)T^n(\tau_1)}\\
\Delta x_{1n} &= x_{1n} - \frac{f(\tau)t(\tau)}{f(\tau_1)(1-\tau)^{\beta}}x_{1n}(0)
\end{aligned}\right\} \tag{137}$$

From expressions (230) and (108), $G_{n,1}(\tau)$ is also of the order $\frac{A_n}{n}t^n$. Thus, the speed of convergence of $x_1(q,\theta)$ and $y_2(q,\theta)$ is the same as that of $\psi_2(q,\theta)$. However, because of the character of $g^{(1)}(\tau)$ as given in table 1 the second series in $x_2(q,\theta)$ and $y_2(q,\theta)$, aside from the factor $\beta\tau$, is generally of superior order as compared with the first one. The smallness of $f^{(1)}(\tau)$ is the main reason, incidently, that the summation generally stops at the first-order terms.

The coordinate functions for the annulus region can similarly be put in the forms:

$$x(q,\theta) = \frac{(1-\tau)^{-\beta}}{q}\left\{\sum_0^{\infty}\left[\frac{\nu B_n}{\nu-1}q^{\nu}F_{\nu}^{(r)}(\tau)\sin(\nu-1)\theta - \frac{\nu C_n}{\nu+1}q^{-\nu}F_{-\nu}^{(r)}(\tau)\sin(\nu+1)\theta\right]\right.$$
$$\left. - \frac{\beta\tau}{2}\sum_0^{\infty}\nu\left[B_n q^{\nu}F_{\nu,1}^{(r)}(\tau) - C_n q^{-\nu}F_{-\nu,1}^{(r)}(\tau)\right]\left[\frac{\sin(\nu-1)\theta}{\nu-1} + \frac{\sin(\nu+1)\theta}{\nu+1}\right]\right\} + x_3 \tag{138}$$

$$y(q,\theta) = \frac{(1-\tau)^{-\beta}}{q}\left\{\sum_{0}^{\infty}\left[\frac{\nu B_n}{\nu - 1} q^{\nu} F_{\nu}^{(r)}(\tau) \cos(\nu - 1)\theta + \frac{\nu C_n}{\nu + 1} q^{-\nu} F_{-\nu}^{(r)}(\tau) \cos(\nu + 1)\theta\right]\right.$$

$$\left. - \frac{\beta\tau}{2}\sum_{0}^{\infty} \nu \left[B_n q^{\nu} F_{\nu,1}^{(r)}(\tau) - C_n q^{-\nu} F_{-\nu,1}^{(r)}(\tau)\right]\left[\frac{\cos(\nu - 1)\theta}{\nu - 1} - \frac{\cos(\nu + 1)\theta}{\nu + 1}\right]\right\} + y_3 \qquad (139)$$

Here $x_3(q,\theta)$ and $y_3(q,\theta)$, corresponding to $\psi_3(q,\theta)$, represent

$$x_3(q,\theta) = -\frac{(1-\tau)^{-\beta}}{2q}\left\{\sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\xi_{-n}(\tau)\left[\frac{\cos(n+1)\theta}{n+1} + \frac{\cos(n-1)\theta}{n-1}\right]\right.$$

$$\left. - \sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\left[\frac{\cos(n+1)\theta}{n+1} - \frac{\cos(n-1)\theta}{n-1}\right] + 2\tilde{D}_0 \cos\theta\right\} + x_{out} \qquad (140)$$

$$y_3(q,\theta) = -\frac{(1-\tau)^{-\beta}}{2q}\left\{\sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\xi_{-n}(\tau)\left[\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n-1)\theta}{n-1}\right]\right.$$

$$\left. - \sum_{1}^{\infty} n\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\left[\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{n-1}\right] + 2\tilde{D}_0 \sin\theta\right\} \qquad (141)$$

Like $\psi_3(q,\theta)$, $x_3(q,\theta)$ and $y_3(q,\theta)$ are also of inferior order and hence further transformation is not required. But, for the other part, as the point $\tau = \frac{1}{2\beta + 1}$ lies in the region $1 < q < V$, it demands separate consideration. First, let it be assumed that the critical speed is less than V; then for the region $1 < q < V$ but $\tau_1 < \tau < \frac{1}{2\beta + 1}$, the following equations are obtained by use of the corresponding functions $x_o(q,\theta)$, $y_o(q,\theta)$, and $\varphi_o(q,\theta)$ of the similar incompressible flow:

$$x(q,\theta) = x_1(q,\theta) + x_2(q,\theta) + x_3(q,\theta) \tag{142}$$

$$y(q,\theta) = y_1(q,\theta) + y_2(q,\theta) + y_3(q,\theta) \tag{143}$$

Here $x_1(q,\theta)$ and $y_1(q,\theta)$ are given respectively by equations (132) and (133), except that $X(q,\theta)$ is replaced by $x(q,\theta) + A_0(\pi - \theta)$, and $x_2(q,\theta)$ and $y_2(q,\theta)$ are given by

$$\begin{aligned} x_2(q,\theta) = \frac{(1-\tau)^{-\beta}}{q} \Bigg\{ & \sum_0^\infty \left[\frac{\nu B_n}{\nu - 1} H_\nu(\tau) q^\nu \sin(\nu - 1)\theta - \frac{\nu C_n}{\nu + 1} H_{-\nu}(\tau) q^{-\nu} \sin(\nu + 1)\theta \right] \\ & - \beta\tau \sum_0^\infty \left[B_n H_{\nu,1}(\tau) q^\nu - C_n H_{-\nu,1}(\tau) q^{-\nu} \right] \left(\sin\nu\theta \cos\theta - \frac{\cos\nu\theta \sin\theta}{\nu} \right) \Bigg\} \end{aligned} \tag{144}$$

$$\begin{aligned} y_2(q,\theta) = \frac{(1-\tau)^{-\beta}}{q} \Bigg\{ & \sum_0^\infty \left[\frac{\nu B_n}{\nu - 1} H_\nu(\tau) q^\nu \cos(\nu - 1)\theta + \frac{\nu C_n}{\nu + 1} H_{-\nu}(\tau) q^{-\nu} \cos(\nu + 1)\theta \right] \\ & - \beta\tau \sum_0^\infty \left[B_n H_{\nu,1}(\tau) q^\nu - C_n H_{-\nu,1}(\tau) q^{-\nu} \right] \left(\sin\nu\theta \sin\theta + \frac{\cos\nu\theta \cos\theta}{\nu} \right) \Bigg\} \end{aligned} \tag{145}$$

where $H_\nu(\tau)$ and $H_{-\nu}(\tau)$ are the same as those defined by equation (118), and $H_{\nu,1}(\tau)$ and $H_{-\nu,1}(\tau)$ denote

$$\left.\begin{aligned} H_{\nu,1}(\tau) &= F_{\nu,1}(\tau)\Delta\frac{\nu^2 F_\nu^{-1}(\tau_1)}{\nu^2 - 1} + \frac{\Delta F_{\nu,1}(\tau)}{f(\tau_1)T^\nu(\tau_1)} \\ H_{-\nu,1}(\tau) &= F_{-\nu,1}\Delta\frac{\nu^2 F_{-\nu}^{-1}(\tau_1)}{\nu^2 - 1} + \frac{\Delta F_{-\nu,1}(\tau)}{f(\tau_1)T^\nu(\tau_1)} \end{aligned}\right\} \tag{146}$$

On the other hand, for $1 < q < V$ but $\frac{1}{2\beta + 1} < \tau < \tau_1 V^2$, the dominant terms of the hypergeometric functions, by expressions (226), (227), (233), and (234), are

$$F_\nu(\tau) \sim f(\tau) T^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right)$$

$$F_{-\nu}(\tau) \sim \frac{1}{2} f(\tau) T^{-\nu} \cos\left(\nu\omega + \frac{\pi}{4}\right)$$

$$F_{\nu,1}(\tau) \sim g(\tau) T^\nu \cos\left(\nu\omega - \mu^o - \frac{\pi}{4}\right)$$

$$F_{-\nu,1}(\tau) \sim \frac{1}{2} g(\tau) T^{-\nu} \cos\left(\nu\omega + \mu^o + \frac{\pi}{4}\right)$$

By substituting these expressions for the corresponding hypergeometric functions and resolving the products such as

$$2 \cos\left(\nu\omega - \frac{\pi}{4}\right) \sin (\nu - 1)\theta = \sin\left[(\nu - 1)\xi - \left(\frac{\pi}{4} - \omega\right)\right] + \sin\left[(\nu + 1)\eta + \left(\frac{\pi}{4} - \omega\right)\right]$$

$$2 \sin \nu\theta \cos\left(\nu\omega - \mu^o - \frac{\pi}{4}\right) = \sin\left(\nu\xi - \mu^o - \frac{\pi}{4}\right) + \sin\left(\nu\eta + \mu^o + \frac{\pi}{4}\right)$$

into sums, a lengthy but straightforward reduction by means of the known identities, for example:

$$x_o(q,\theta) = \sum_0^\infty \left[\frac{\nu B_n}{\nu - 1} q^{\nu-1} \sin (\nu - 1)\theta - \frac{\nu C_n}{\nu + 1} q^{-\nu-1} \sin (\nu + 1)\theta\right]$$

$$y_o(q,\theta) = \sum_0^\infty \left[\frac{\nu B_n}{\nu - 1} q^{\nu-1} \cos (\nu - 1)\theta + \frac{\nu C_n}{\nu + 1} q^{-\nu-1} \cos (\nu + 1)\theta\right]$$

gives

$$x(q,\theta) = x_1(q,\theta) + x_2(q,\theta) + x_3(q,\theta) \tag{147}$$

$$y(q,\theta) = y_1(q,\theta) + y_2(q,\theta) + y_3(q,\theta) \tag{148}$$

Here $x_3(q,\theta)$ **and** $y_3(q,\theta)$ retain their definitions, and

$$\begin{aligned} x_1(q,\theta) = \frac{(1-\tau)^{-\beta} f(\tau) t(\tau)}{4f(\tau_1)} \Bigg\{ & \Big[x_o(\lambda,\xi) + x_o(\lambda,\eta)\Big] \cos\left(\frac{\pi}{4} - \omega\right) - \Big[y_o(\lambda,\xi) - y_o(\lambda,\eta)\Big] \sin\left(\frac{\pi}{4} - \omega\right) \\ & - \frac{\beta\tau}{\lambda} \frac{g(\tau)}{f(\tau_1)} \Big[\Phi(q,\theta) \cos\theta - \Psi(q,\theta) \sin\theta\Big] \Bigg\} \end{aligned} \tag{149}$$

$$\begin{aligned} y_1(q,\theta) = \frac{(1-\tau)^{-\beta} f(\tau) t(\tau)}{4f(\tau_1)} \Bigg\{ & \Big[y_o(\lambda,\xi) + y_o(\lambda,\eta)\Big] \cos\left(\frac{\pi}{4} - \omega\right) + \Big[x_o(\lambda,\xi) - x_o(\lambda,\eta)\Big] \sin\left(\frac{\pi}{4} - \omega\right) \\ & - \frac{\beta\tau}{\lambda} \frac{g(\tau)}{f(\tau_1)} \Big[\Phi(q,\theta) \sin\theta + \Psi(q,\theta) \cos\theta\Big] \Bigg\} \end{aligned} \tag{150}$$

where

$$\Phi(q,\theta) = \Big[\varphi_o(\lambda,\xi) + \varphi_o(\lambda,\eta)\Big] \cos\left(\mu^o + \frac{\pi}{4}\right) - \Big[\psi_o(\lambda,\xi) - \psi_o(\lambda,\eta)\Big] \sin\left(\mu^o + \frac{\pi}{4}\right) \tag{151}$$

$$\begin{aligned} \Psi(q,\theta) = & \Big[\chi(\lambda,\xi) + \chi(\lambda,\eta) + 2A_o(\pi - \theta)\Big] \cos\left(\mu^o + \frac{\pi}{4}\right) + \Big[\sigma(\lambda,\xi) \\ & - \sigma(\lambda,\eta)\Big] \sin\left(\mu^o + \frac{\pi}{4}\right) \end{aligned} \tag{152}$$

Again $x_2(q,\theta)$ and $y_2(q,\theta)$ are defined by equations (144) and (145), but the differences of the hypergeometric functions involved in $H_{\pm\nu}(\tau)$ and $H_{\pm\nu,1}(\tau)$ are now

$$\left.\begin{aligned}\Delta F_{\pm\nu}(\tau) &= F_{\pm\nu}(\tau) - \frac{f(\tau)}{2} T^{\pm\nu} \cos\left(\nu\omega \mp \frac{\pi}{4}\right)\\ \Delta F_{\pm\nu,1}(\tau) &= F_{\pm\nu,1}(\tau) - \frac{g(\tau)}{2} T^{\pm\nu} \cos\left(\nu\omega \mp \mu^{\rho} \mp \frac{\pi}{4}\right)\end{aligned}\right\} \quad \frac{1}{2\beta + 1} < \tau < 1$$

Improvement of Convergence of $\tilde{A}_n$, $\tilde{D}_n$, and x_{1n}

As seen from part II, as soon as the function $\psi_o(q,\theta)$ is specified, the coefficients of the inside and outside series can be chosen so as to make the conditions of continuity sufficient for the sets of unknown constants, for example, $\tilde{A}_n$ and $\tilde{D}_n$, to be determined. Inasmuch as the conditions are applied at the circle of convergence, it is generally very tedious to evaluate $\tilde{A}_n$ and $\tilde{D}_n$ from the slowly convergent series (equations (54) and (55)) when B_n and C_n are given. In order to bring these expressions to manageable forms, the following procedure can be used.

By considering the function $\psi_o(q,\theta)$ the following identities (appendix F) for $n > 1$ can be deduced without difficulty

$$A_n = \frac{1}{\pi} \sum_{0}^{\infty} (B_m + C_m) I_{n\mu} \tag{153}$$

$$nA_n = \frac{1}{\pi} \sum_{0}^{\infty} \mu(B_m - C_m) I_{n\mu} \tag{154}$$

$$\frac{A_n}{n} = \frac{1}{\pi} \sum_{0}^{\infty} \frac{1}{\mu}(B_m - C_m) I_{n\mu} + \frac{4}{n} \tag{155}$$

Furthermore, the sum on the right-hand side of the second expression of equation (52) can be written as

$$\frac{1}{\pi}\sum_{0}^{\infty}\mu\Big[B_m\xi_\mu(\tau_1) + C_m\xi_{-\mu}(\tau_1)\Big]I_{n\mu} = \frac{1}{\pi}\sum_{0}^{\infty}\mu\Big[B_m\tilde{\xi}_\mu(\tau_1) + C_m\tilde{\xi}_{-\mu}(\tau_1)\Big]I_{n\mu}$$

$$+ \frac{1}{\pi}\sum_{0}^{\infty}\mu\Big[B_m\,\Delta\xi_\mu(\tau_1) + C_m\,\Delta\xi_{-\mu}(\tau_1)\Big]I_{n\mu}$$

where, according to expressions (243) and (244), $\tilde{\xi}_{\pm\mu}(\tau_1) = \pm\xi_o(\tau_1) + \frac{\xi^{(1)}(\tau)}{\mu} \pm \frac{\xi^{(2)}(\tau_1)}{\mu^2}$ and

$$\Delta\xi_{\pm\mu}(\tau_1) = \tilde{\xi}_{\pm\mu}(\tau_1) - \tilde{\xi}_{\pm\mu}(\tau_1) \tag{156}$$

Then from equations (153), (154), and (155) it follows that

$$\frac{1}{\pi}\sum_{0}^{\infty}\mu\Big[B_m\xi_\mu(\tau_1) + C_m\xi_{-\mu}(\tau_1)\Big]I_{n\mu} = nA_n\tilde{\xi}_n(\tau_1) - \frac{4\xi^{(2)}(\tau_1)}{n} + \sigma_n \tag{157}$$

where

$$\sigma_n = \frac{1}{\pi}\sum_{0}^{\infty}\mu\Big[B_m\Delta\xi_\mu(\tau_1) + C_m\Delta\xi_{-\mu}(\tau_1)\Big]I_{n\mu} \tag{158}$$

Now by expressions (243) and (244) $\Delta\xi_{\pm\mu} \sim \mu^{-3}$; σ_n converges like $\mu^{-7/2}$ if C_m is of the order $m^{-1/2}$. From equation (157), equations (54) and (55) become

$$\tilde{A}_n = \frac{1}{\xi_n(\tau_1) - \xi_{-n}(\tau_1)}\left\{\Big[\tilde{\xi}_n(\tau_1) - \xi_{-n}(\tau_1)\Big]A_n - \frac{1}{n}\Big[2\tilde{D}_o\xi_{-n}(\tau_1) - \sigma_n\Big] - \frac{4\xi^{(2)}(\tau_1)}{n^2}\right\} \tag{159}$$

$$\tilde{D}_n = \frac{1}{\xi_n(\tau_1) - \xi_{-n}(\tau_1)}\left\{\frac{1}{n}\left[2\tilde{D}_o\xi_n(\tau_1) - \sigma_n\right] + \frac{4\xi^{(2)}(\tau_1)}{n^2} + A_n\Delta\xi_n(\tau_1)\right\} \tag{160}$$

Another important constant in connection with the present problem is the major axis x_{in}, or the half-chord of the body. As shown in equation (95) it was also given in terms of slowly convergent series. In order to transform the series, it should be recalled that for the similar incompressible flow the half-chord is

$$x_{in}(o) = \frac{1}{2\pi}\left\{\sum_0^\infty \nu(B_n - C_n)\left[\frac{1}{(\nu-1)^2} + \frac{1}{(\nu+1)^2}\right] + \sum_0^\infty \nu(B_n + C_n)\left[\frac{1}{(\nu-1)^2} - \frac{1}{(\nu+1)^2}\right]\right\} \tag{161}$$

Then, by replacing $\xi_{\pm\nu}(\tau_1)$ by $\pm 1 \mp \beta\tau\dfrac{F_{\pm\nu,1}(\tau_1)}{F_{\pm\nu}(\tau_1)}$, respectively, it follows in the first place that

$$x_{in} = (1-\tau_1)^{-\beta}\left\{x_{in}(o) - \frac{\beta\tau_1}{2\pi}\sum_0^\infty \nu\left[B_nF_{\nu,1}{}^{(r)}(\tau_1) - C_nF_{-\nu,1}{}^{(r)}\right]\left[\frac{1}{(\nu-1)^2} + \frac{1}{(\nu+1)^2}\right]\right\} + x_{out} \tag{162}$$

Furthermore, by writing

$$\sum_0^\infty \nu\left[B_nF_{\nu,1}{}^{(r)} - C_nF_{-\nu,1}{}^{(r)}\right]\left[\frac{1}{(\nu-1)^2} + \frac{1}{(\nu+1)^2}\right] = \frac{g(\tau_1)}{f(\tau_1)}\sum_0^\infty \nu(B_n - C_n)\left[\frac{1}{(\nu-1)^2} + \frac{1}{(\nu+1)^2}\right]$$
$$+ \sum_0^\infty \nu\left[B_n\Delta F_{\nu,1}{}^{(r)} - C_n\Delta F_{-\nu,1}{}^{(r)}\right]\left[\frac{1}{(\nu-1)^2} + \frac{1}{(\nu+1)^2}\right]$$

where

$$\Delta F_{\nu,1}^{(r)}(\tau_1) = \frac{F_{\nu,1}(\tau_1)}{F_\nu(\tau_1)} - \frac{g(\tau_1)}{f(\tau_1)}$$

$$\Delta F_{-\nu,1}^{(r)}(\tau_1) = \frac{F_{-\nu,1}(\tau_1)}{F_{-\nu}(\tau_1)} - \frac{g(\tau_1)}{f(\tau_1)}$$

with equation (161), the following equation is obtained:

$$x_{in} = (1 - \tau_1)^{-\beta} \left\{ x_{in}(o)\left[1 - \frac{\beta\tau_1 g(\tau_1)}{f(\tau_1)}\right] + \frac{\beta\tau_1}{2\pi} \frac{g(\tau_1)}{f(\tau_1)} \sum_0^\infty \nu(B_n + C_n)\left[\frac{1}{(\nu - 1)^2} - \frac{1}{(\nu + 1)^2}\right] \right.$$
$$\left. - \sum_0^\infty \nu\left[B_n \Delta F_{\nu,1}^{(r)} - C_n \Delta F_{-\nu,1}^{(r)}\right]\left[\frac{1}{(\nu - 1)^2} + \frac{1}{(\nu + 1)^2}\right] \right\} + x_{out} \qquad (163)$$

Since $\Delta F_{\pm\nu,1}^{(r)}(\tau_1) = 0\left(\frac{1}{\nu}\right)$, it is easy to see that both series now converge as $\nu^{-5/2}$. Because of the fact that $\beta\tau_1 << 1$, however, the contribution due to both series is generally small.

IV - APPLICATION TO CASE OF ELLIPTIC CYLINDER

The Functions $z_o(w)$, $W_o(w)$, and $\Lambda(w)$

An irrotational flow of an incompressible fluid about an elliptic cylinder with a circulation Γ_o is represented by the complex potential $W_o(z_o)$.

$$W_o(z_o) = \zeta + \frac{1}{\zeta} + \frac{i\Gamma_o}{2\pi} \log_e \zeta \qquad \text{with} \qquad z_o = \zeta + \frac{\epsilon^2}{\zeta} \tag{164}$$

where the flow at infinity is assumed to be parallel to the major axis of the cylinder. Here all the quantities have been rendered dimensionless by normalizing ζ by a length a and $W_o(z_o)$, by Ua. Then the major and minor axes are, respectively, $1 + \epsilon^2$ and $1 - \epsilon^2$, where $\epsilon^2 < 1$.

By differentiating equation (164) with respect to z_o, the dimensionless complex velocity of the flow is obtained, namely,

$$w = \frac{\zeta^2 + \frac{i\Gamma_o}{2\pi}\zeta - 1}{\zeta^2 - \epsilon^2}$$

the inverse solution of which is

$$\zeta(w) = -\frac{\frac{i\Gamma_o}{2\pi} + \left[4(1 - w)(1 - \epsilon^2 w) - \frac{\Gamma_o^2}{4\pi^2}\right]^{1/2}}{2(1 - w)} \tag{165}$$

This function is two-valued with two simple branch points at $w = 1 + O\left(\Gamma_o^2\right)$ and $w = \epsilon^{-2} + O\left(\Gamma_o^2\right)$, namely, $V = \epsilon^{-2} + O\left(\Gamma_o^2\right)$. (See section entitled "Analytic Continuation of Solution (Branch Point of Order 1).") The principal value may be defined by the convention that $-\pi < \arg(1 - w) < \pi$ and $1 < |w| < \epsilon^{-2}$ with a cut joining the two branch points. With the principal value so chosen, the domain wherein the real part of z_o is less than or equal to 0, excluding the interior of the body, corresponds uniquely to the hodograph given by $W(z_o)$ or, conversely, on account of symmetry, the domain wherein the real part of z_o is greater than or equal to 0 will correspond to the second branch of the function $\zeta(w)$.

By substituting $\zeta(w)$ in equations (165) and (164), an expansion regarding $\frac{\Gamma_o}{4\pi}$ as a small parameter gives in accordance with equations (35) and (36), provided that $\frac{\Gamma_o}{4\pi} << 1$,

$$z_o^{(o)}(w) = -\left[\left(\frac{1-\epsilon^2 w}{1-w}\right)^{1/2} + \epsilon^2\left(\frac{1-w}{1-\epsilon^2 w}\right)^{1/2}\right] \tag{166}$$

$$W_o^{(o)}(w) = -\left[\left(\frac{1-\epsilon^2 w}{1-w}\right)^{1/2} + \left(\frac{1-w}{1-\epsilon^2 w}\right)^{1/2}\right] \tag{167}$$

$$z_o^{(1)}(w) = -i\left(\frac{1}{1-w} - \frac{\epsilon^2}{1-\epsilon^2 w}\right) \tag{168}$$

$$W_o^{(1)}(w) = -i\left[\frac{1}{1-w} + \log_e(1-w) - \frac{1}{1-\epsilon^2 w} - \log_e(1-\epsilon^2 w) - 2\pi i\right] \tag{169}$$

The functions $z_o^{(o)}(w)$ and $W_o^{(o)}(w)$ are the transition function and the complex potential, respectively, for zero circulation (compare reference 1); and $z_o^{(1)}(w)$ and $W_o^{(1)}(w)$ represent the first-order contributions, due to circulation, to $z_o(w)$ and $W_o(w)$, respectively. On separating into real and imaginary parts, it was found that from $z_o^{(o)}(w)$ and $W_o^{(o)}(w)$ there result

$$x_o^{(o)}(q,\theta) = -\frac{1}{\sqrt{2}}\left\{\left[I(q,\theta) + J(q,\theta)\right]^{1/2} + \epsilon^2\left[I_\epsilon + J^{-1}(q,\theta)\right]^{1/2}\right\} \tag{170}$$

$$y_o^{(o)}(q,\theta) = \frac{1}{\sqrt{2}}\left\{\left[-I(q,\theta) + J(q,\theta)\right]^{1/2} - \epsilon^2\left[-I_\epsilon + J^{-1}(q,\theta)\right]^{1/2}\right\} \tag{171}$$

and

$$\varphi_o^{(o)}(q,\theta) = -\frac{1}{\sqrt{2}}\left\{\left[I(q,\theta) + J(q,\theta)\right]^{1/2} + \left[I_\epsilon + J^{-1}(q,\theta)\right]^{1/2}\right\} \tag{172}$$

$$\psi_o^{(o)}(q,\theta) = \frac{1}{\sqrt{2}}\left\{\left[-I(q,\theta) + J(q,\theta)\right]^{1/2} - \left[-I_\epsilon(q,\theta) + J^{-1}(q,\theta)\right]^{1/2}\right\} \tag{173}$$

where

$$I(q,\theta) = \frac{1-(1+\epsilon^2)q\cos\theta + \epsilon^2 q^2}{1-2q\cos\theta + q^2} \tag{174}$$

$$I_\epsilon(q,\theta) = \frac{1 - (1 + \epsilon^2)q \cos\theta + \epsilon^2 q^2}{1 - 2\epsilon^2 q \cos\theta + \epsilon^4 q^2} \tag{175}$$

$$J(q,\theta) = \left(\frac{1 - 2\epsilon^2 q \cos\theta + \epsilon^4 q^2}{1 - 2q\cos\theta + q^2}\right)^{1/2} \tag{176}$$

Similarly, from $z_o^{(1)}(w)$ and $W_o^{(1)}(w)$,

$$x_o^{(1)}(q,\theta) = -\frac{q \sin\theta}{1 - 2q\cos\theta + q^2} + \frac{\epsilon^4 q \sin\theta}{1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2} \tag{177}$$

$$y_o^{(1)}(q,\theta) = -\frac{1 - q\cos\theta}{1 - 2q\cos\theta + q^2} + \frac{\epsilon^2(1 - \epsilon^2 q\cos\theta)}{1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2} \tag{178}$$

and, if $q < 1$,

$$\left.\begin{aligned}
\varphi_o^{(1)}(q,\theta) &= -\frac{q\sin\theta}{1 - 2q\cos\theta + q^2} + \frac{\epsilon^2 q\sin\theta}{1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2} + \tan^{-1}\frac{q\sin\theta}{1 - q\cos\theta} \\
&\quad - \tan^{-1}\frac{\epsilon^2 q\sin\theta}{1 - \epsilon^2 q\cos\theta} - 2\pi \\
&\text{or, if } q > 1, \\
\varphi_o^{(1)}(q,\theta) &= -\frac{q\sin\theta}{1 - 2q\cos\theta + q^2} + \frac{\epsilon^2 q\sin\theta}{1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2} + \tan^{-1}\frac{q^{-1}\sin\theta}{1 - q^{-1}\cos\theta} \\
&\quad - \tan^{-1}\frac{\epsilon^2 q\sin\theta}{1 - \epsilon^2 q\cos\theta} - (\pi - \theta)
\end{aligned}\right\} \tag{179}$$

$$\psi_o^{(1)}(q,\theta) = -\frac{1 - q\cos\theta}{1 - 2q\cos\theta + q^2} + \frac{1 - \epsilon^2 q\cos\theta}{1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2} - \frac{1}{2}\log_e\left(1 - 2q\cos\theta + q^2\right)$$

$$+ \frac{1}{2}\log_e\left(1 - 2\epsilon^2 q\cos\theta + \epsilon^4 q^2\right) \tag{180}$$

From appendix F, the function $\Lambda(w)$ is defined by

$$\Lambda(w) = \int_0^w \left[W_o^{(o)}(w) + A_o\right]\frac{dw}{w}$$

where the constant A_o is 2, from equation (172). By substituting the function $W_o^{(o)}(w)$ from equation (167), the integration yields

$$\Lambda(w) = -\frac{2(1 + \epsilon^2)}{\epsilon}\log_e\frac{(1 - \epsilon^2 w)^{1/2} + \epsilon(1 - w)^{1/2}}{1 + \epsilon} + 4\log_e\frac{(1 - \epsilon^2 w)^{1/2} + (1 - w)^{1/2}}{2} \tag{181}$$

On separating again into real and imaginary parts, it was found that

$$\sigma(q,\theta) = -\frac{1 + \epsilon^2}{\epsilon}\log_e\frac{1}{2(1 + \epsilon)^2}\left[(K_\epsilon + \epsilon K)^2 + (\tilde{K}_\epsilon + \epsilon\tilde{K})^2\right]$$

$$+ 2\log_e\frac{1}{8}\left[(K_\epsilon + K)^2 + (\tilde{K}_\epsilon + \tilde{K})^2\right] \tag{182}$$

$$\chi(q,\theta) = -\frac{2(1 + \epsilon^2)}{\epsilon}\tan^{-1}\frac{\tilde{K}_\epsilon + \epsilon\tilde{K}}{K_\epsilon + \epsilon K} + 4\tan^{-1}\frac{\tilde{K}_\epsilon + \tilde{K}}{K_\epsilon + K} \tag{183}$$

where

$$K(q,\theta) = \left[(1 - 2q\cos\theta + q^2)^{1/2} + 1 - q\cos\theta\right]^{1/2} \tag{184}$$

$$K_\epsilon(q,\theta) = \left[(1 - 2\epsilon^2 q \cos\theta + \epsilon^4 q^2)^{1/2} + 1 - \epsilon^2 q \cos\theta\right]^{1/2} \tag{185}$$

$$\bar{K}(q,\theta) = \left[(1 - 2q \cos\theta + q^2)^{1/2} - 1 + q \cos\theta\right]^{1/2} \tag{186}$$

$$\bar{K}_\epsilon(q,\theta) = \left[(1 - 2\epsilon^2 q \cos\theta + \epsilon^4 q^2)^{1/2} - 1 + \epsilon^2 q \cos\theta\right]^{1/2} \tag{187}$$

Expansions of $W_o(w)$ and $z_o(w)$

As the domain considered is complex and has two singularities at $w = 1$ and $w = \epsilon^{-2}$, the function $W_o^{(o)}(w)$ defined by equation (167) is single-valued and continuous in $|w| < 1$ but is discontinuous across the cut in $1 < |w| < \epsilon^{-2}$. The expansions, as shown in reference 8, are

$$W_o^{(o)}(w) = -\sum_0^\infty A_n w^n \qquad |w| < 1 \tag{188}$$

and

$$W_o^{(o)}(w) = i\sum_0^\infty (B_n w^\nu + C_n w^{-\nu}) \qquad 1 < |w| < \epsilon^{-2} \tag{189}$$

where the coefficients are all real and are defined by

$$A_n = 2S_n^{(i)} - (1 + \epsilon^2)S_{n-1}^{(i)} \qquad n > 1 \tag{190}$$

$$A_o = 2S_o^{(i)} = 2$$

$$A_1 = 0$$

with

$$S_n^{(i)} = \frac{1}{\pi}\sum_0^n \frac{\Gamma\left(n - m + \frac{1}{2}\right)\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(n - m + 1)\Gamma(m + 1)} \epsilon^{2m} \tag{191}$$

and

$$B_n = \left[2\epsilon^2 S_{n+1}^{(o)} - (1 + \epsilon^2)S_n^{(o)}\right]\epsilon^{2n} \tag{192}$$

$$C_n = 2S_n^{(o)} - (1 + \epsilon^2)S_{n+1}^{(o)} \tag{193}$$

with

$$S_n^{(o)} = \frac{1}{\pi}\sum_{0}^{\infty} \frac{\Gamma\left(n + m + \frac{1}{2}\right)\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(n + m + 1)\Gamma(m + 1)} \epsilon^{2m} \tag{194}$$

Similarly, the function $W_o^{(1)}(w)$, as defined by equation (169), can be easily expanded. It is, for $|w| < 1$,

$$W_o^{(1)}(w) = i\left(-\sum_{0}^{\infty} w^n + \sum_{1}^{\infty} \frac{w^n}{n} + \sum_{0}^{\infty} \epsilon^{2n}w^n - \sum_{1}^{\infty} \frac{\epsilon^{2n}w^n}{n}\right) - 2\pi \tag{195}$$

and, for $1 < |w| < \epsilon^{-2}$,

$$W_o^{(1)}(w) = i\left[\sum_{1}^{\infty} w^{-n} + \sum_{1}^{\infty} \frac{w^{-n}}{n} + \sum_{0}^{\infty} \epsilon^{2n}w^n - \sum_{1}^{\infty} \frac{\epsilon^{2n}w^n}{n} - \log_e q + i(\pi + \theta)\right] \tag{196}$$

By separating into real and imaginary parts, the following equations are obtained from $W_o^{(o)}(w)$:

$$\varphi_o^{(o)}(q,\theta) = -\sum_{2}^{\infty} A_n q^n \cos n\theta - 2 \tag{197}$$

$$q < 1$$

$$\psi_o^{(o)}(q,\theta) = \sum_{2}^{\infty} A_n q^n \sin n\theta \tag{198}$$

and

$$\varphi_o^{(o)}(q,\theta) = \sum_{0}^{\infty} \left(B_n q^{\nu} - C_n q^{-\nu}\right) \sin \nu\theta \tag{199}$$

$$\psi_o^{(o)}(q,\theta) = \sum_0^\infty (B_n q^\nu + C_n q^{-\nu}) \cos \nu\theta \qquad 1 < q < \epsilon^{-2} \tag{200}$$

Similarly, from $W_o^{(1)}(w)$, the result is

$$\varphi_o^{(1)}(q,\theta) = -\sum_2^\infty A_n^{(1)} q^n \sin n\theta - 2\pi \tag{201}$$

$$q < 1$$

$$\psi_o^{(1)}(q,\theta) = -\sum_2^\infty A_n^{(1)} q^n \cos n\theta \tag{202}$$

where the coefficient $A_n^{(1)}$ is given by

$$A_n^{(1)} = \left(1 - \frac{1}{n}\right)\left(1 - \epsilon^{2n}\right) \tag{203}$$

and

$$\varphi_o^{(1)}(q,\theta) = \sum_1^\infty \left[B_n^{(1)} q^n - C_n^{(1)} q^{-n}\right] \sin n\theta - (\pi + \theta) \tag{204}$$

$$\psi_o^{(1)}(q,\theta) = \sum_1^\infty \left[B_n^{(1)} q^n + C_n^{(1)} q^{-n}\right] \cos n\theta - \log_e q + 1 \tag{205}$$

where

$$B_n^{(1)} = \left(1 - \frac{1}{n}\right)\epsilon^{2n} \tag{206}$$

$$C_n^{(1)} = 1 + \frac{1}{n} \tag{207}$$

The expansion of $z_o(w)$ can be carried out in like manner, but it was found simpler to deduce it directly from equation (37). Then, corresponding to $W_o^{(o)}(w)$ defined by equations (188) and (189), $z_o^{(o)}(w)$ is

$$z_o^{(o)}(w) = -\sum_2^\infty \frac{nA_n}{n - 1} w^n - (1 + \epsilon^2) \qquad |w| < 1 \tag{208}$$

and

$$z_o^{(o)}(w) = i\sum_{0}^{\infty}\left(\frac{\nu B_n}{\nu - 1}w^{\nu-1} + \frac{\nu C_n}{\nu + 1}w^{-\nu-1}\right) \qquad 1 < |w| < \epsilon^{-2} \tag{209}$$

In determining the constants of integration, use has been made of the fact that $z_o^{(o)}(o) = -(1 + \epsilon^2)$ from equation (77) and that $x_o^{(o)}(q,\theta) = 0$ and $y_o^{(o)}(q,\theta) = -y_o^{(o)}(q,2\pi - \theta)$.

On the other hand, corresponding to $W_o^{(1)}(w)$, $z_o^{(1)}(w)$ is

$$z_o^{(1)}(w) = -i\sum_{2}^{\infty}\frac{nA_n^{(1)}}{n - 1}w^{n-1} - i(1 - \epsilon^2) \qquad |w| < 1 \tag{210}$$

and

$$z_o^{(1)}(w) = i\sum_{1}^{\infty}\left[\frac{nB_n^{(1)}}{n - 1}w^{n-1} + \frac{nC_n^{(1)}}{n + 1}w^{-n-1}\right] + \frac{i}{w} \qquad 1 < |w| < \epsilon^{-2} \tag{211}$$

Here again the constants are determined with the aid of equation (168). By separating into real and imaginary parts, the coordinate functions are, for $q < 1$,

$$x_o^{(1)}(q,\theta) = -\sum_{2}^{\infty}\frac{nA_n}{n - 1}q^{n-1}\cos(n - 1)\theta - (1 + \epsilon^2) \tag{212}$$

$$y_o^{(1)}(q,\theta) = \sum_{2}^{\infty}\frac{nA_n}{n - 1}q^{n-1}\sin(n - 1)\theta \tag{213}$$

and, for $1 < q < \epsilon^{-2}$,

$$x_o^{(o)}(q,\theta) = -\sum_{0}^{\infty}\left[\frac{\nu B_n}{\nu - 1}q^{\nu-1}\sin(\nu - 1)\theta - \frac{\nu C_n}{\nu + 1}q^{-\nu-1}\sin(\nu + 1)\theta\right] \tag{214}$$

$$y_o^{(o)}(q,\theta) = \sum_0^{\infty} \left[\frac{\nu B_n}{\nu - 1} q^{\nu-1} \cos (\nu - 1)\theta + \frac{\nu C_n}{\nu + 1} q^{-\nu-1} \cos (\nu + 1)\theta \right] \tag{215}$$

Similarly, from equations (210) and (211), for $q < 1$,

$$x_o^{(1)}(q,\theta) = -\sum_2^{\infty} \frac{n}{n - 1} A_n^{(1)} q^{n-1} \sin (n - 1)\theta \tag{216}$$

$$y_o^{(1)}(q,\theta) = -\sum_2^{\infty} \frac{n}{n - 1} A_n^{(1)} q^{n-1} \cos (n - 1)\theta - (1 - \epsilon^2) \tag{217}$$

and, for $1 < q < \epsilon^{-2}$,

$$x_o^{(1)}(q,\theta) = \sum_1^{\infty} \left[\frac{nB_n^{(1)}}{n - 1} q^{n-1} \sin (n - 1)\theta - \frac{nC_n^{(1)}}{n + 1} q^{-n-1} \sin (n + 1)\theta \right] - \frac{\sin \theta}{q} \tag{218}$$

$$y_o^{(1)}(q,\theta) = \sum_1^{\infty} \left[\frac{nB_n^{(1)}}{n - 1} q^{n-1} \cos (n - 1)\theta + \frac{nC_n^{(1)}}{n + 1} q^{-n-1} \cos (n + 1)\theta \right] + \frac{\cos \theta}{q} \tag{219}$$

where the first term in equation (219) gives rise to a constant ϵ^2 due to the definition of $B_n^{(1)}$.

Numerical Results and Discussion

In order to conclude this report, two numerical examples are presented for the purpose of testing the effectiveness of the method, of giving a general idea of the result that might be expected from a given similar incompressible flow, and, as far as possible, of evaluating the compressibility effects.

By taking $\epsilon = \frac{1}{2}$ and $M_1 = 0.60$, M_1 being the free-stream Mach number, the compressible flow without circulation is shown in figures 2 and 3 in both the τ,θ- and the x,y-planes. The flow is everywhere continuous and the highest Mach number attained at the central section is 1.24. The profile calculated in this case is symmetrical but is far different from the ellipse from which it was derived. The thickness ratio is 1:2 as compared with 3:5 of the original elliptic section. The lateral distortion thus is much more pronounced than that suffered by the longitudinal dimension. The characteristic feature in the case of transonic flows is that the central portion of the derived body where the flow is supersonic is always flat in comparison with the original body.

In order to exhibit the compressiblity effect, the corresponding problem of the incompressible flow over the same body must be solved. Technically, this does not offer any difficulty. However, in order to simplify the numerical work, an incompressible flow resulting from a superposition of a parallel flow on a source-sink combination has been considered. It gives rise to a body which, by adjusting the strength of and the distance between the source and sink, approximates closely the given body with an error of about 2 or 3 percent. (Compare fig. 3.) The pressure distributions over the same body for both compressible and incompressible fluids are compared in figure 4. The results calculated according to the Von Kármán-Tsien and the Glauert-Prandtl formulas are also given.

The wide disagreement between the exact and approximate curves serves as a proof that in the case of transonic flows the thickness effect of the body is no longer secondary and cannot be ignored entirely, as is done in both approximate formulas. The immediate cause for these discrepancies seems to be the fact that the point of equal pressure on the surface of the body in the case of compressible and incompressible fluids does not correspond with that of the free-stream pressure. This deviation has, in fact, been observed even in the case of subsonic flow about a Joukowski airfoil of thickness ratio 0.15 (reference 9). The reason is that the flow of a compressible fluid over a body is partly compressive and partly expansive. By increasing the free-stream Mach number, the speed of the flow in the neighborhood of the stagnation point tends to decrease and the speed far away from the stagnation point, that is, in the supersonic region, to increase. The hodograph corresponding to the zero streamline then becomes longer and flatter as the free-stream Mach number increases. Consequently, the point of intersection of the compressible and incompressible hodographs will shift toward the small inclination of the velocity vector, that is, away from the stagnation point. This effect will be more pronounced for larger thickness ratio as well as Mach number.

The general applicability of the Von Kármán-Tsien formula has been questioned and examined by Tsien and A. Fejer (reference 10) from purely geometric considerations, and the limitations therein raised seem to be further substantiated by the present result. The question, however, remains to be answered as to whether critical conditions for

the validity of the Von Kármán–Tsien theory could be given in terms of the thickness ratio and Mach number. From the practical point of view establishment of such conditions would constitute a result of major importance.

For the case of circulatory flow, $\frac{\Gamma_o}{4\pi}$ is taken to be 0.05. Here, for simplicity's sake, $A_o^{(1)} = 1$ and $B_o^{(1)} = (1 - \tau_1)^{-\beta}$ by equation (69), so that the circulation of the compressible flow is $(1 - \tau_1)^{-\beta}\Gamma_o$, according to equation (74). The constant $C_o^{(1)}$ could have been determined by successive approximations, but in this particular case it is small enough to be neglected. The calculated flow pattern is shown in figures 5 and 6 for both planes. The body, as expected, is unsymmetrical with a negative camber of about 2 percent. The highest Mach number reached at the upper surface is 1.33 and that at the lower surface is 1.15.

The lift coefficient is 0.65 corresponding to an angle of attack at zero lift of about 2.2°. For comparsion with the case of incompressible flow, the problem is simplified by considering the incompressible flow over the body resulting from the superposition of a small negative camber on a geometrically similar ellipse of thickness ratio 1:2. The estimated lift coefficient of such a system is approximately 0.45. The ratio of the two coefficients thus is nearly 1.5, which is much higher than the value of 1.25 given by the Glauert–Prandtl formula for the present case. Actually, if the oval shape is considered, the incompressible value of the lift coefficient would have been lower than that obtained, owing to the fact that the central flat portion reduces the speed of the flow and hence leads to a higher pressure. Thus, the ratio of about 1.5 should be regarded as a lower limit.

Guggenheim Aeronautical Laboratory
California Institute of Technology
Pasadina, Calif., September 5, 1946

APPENDIX A

PARTICULAR INTEGRALS OF $\varphi(q,\theta)$

Consider the total differential $d\varphi$ which from equations (10) and (11) is

$$d\varphi = -\frac{\rho_o}{\rho}\left(1 - M^2\right)\frac{1}{q}\psi_\theta\, dq + \frac{\rho_o}{\rho} q\psi_q\, d\theta$$

If the particular integrals of ψ are given by the first group of expression (24), then

$$d\varphi = -\nu\frac{\rho_o}{\rho}\left(1 - M^2\right)\frac{\psi_\nu(q)}{q}\begin{Bmatrix} -\sin\nu\theta \\ \cos\nu\theta \end{Bmatrix} dq$$

$$+ \frac{\rho_o}{\rho} q \frac{d}{dq}\left[q^\nu F_\nu(\tau)\right]\begin{Bmatrix} \cos\nu\theta \\ \sin\nu\theta \end{Bmatrix} d\theta$$

From equation (13) $\psi_\nu(q)$ can be eliminated as follows:

$$d\varphi = \frac{\partial}{\partial q}\left[\frac{\rho_o}{\rho} q^\nu F_\nu(\tau)\xi_\nu(\tau)\right]\begin{Bmatrix} \sin\nu\theta \\ -\cos\nu\theta \end{Bmatrix} dq$$

$$+ \frac{\partial}{\partial\theta}\left[\frac{\rho_o}{\rho} q^\nu F_\nu(\tau)\xi_\nu(\tau)\right]\begin{Bmatrix} \sin\nu\theta \\ -\cos\nu\theta \end{Bmatrix} d\theta$$

The integration therefore gives

$$\varphi = \frac{\rho_o}{\rho} q^\nu F_\nu(\tau)\xi_\nu(\tau)\begin{Bmatrix} \sin\nu\theta \\ -\cos\nu\theta \end{Bmatrix} + \text{Constant} \tag{220}$$

Similarly, the following equation corresponding to the second group of expression (24) is obtained:

$$\varphi = \frac{\rho_o}{\rho} q^{-\nu} F_{-\nu}(\tau)\xi_{-\nu}(\tau)\begin{Bmatrix} \sin\nu\theta \\ -\cos\nu\theta \end{Bmatrix} + \text{Constant} \tag{221}$$

APPENDIX B

PROOF OF THEOREM 1

Consider a small neighborhood of $q = 0$ where the hypergeometric functions can be accurately represented by

$$F_n(\tau) = f(\tau)T^n(\tau)\left[1 + \frac{f^{(1)}(\tau)}{n}\right]$$

If the complex potential for the similar incompressible flow is

$$W_o(w) = -\sum_0^\infty A_n w^n \qquad |w| < 1$$

let $W(w,\tau)$ be a new complex potential such that the stream function for the compressible flow is given by

$$\psi(q,\theta) = I_m\left[W(w,\tau)\right]$$

Then $W(w,\tau)$ must be of the form

$$W(w,\tau) = -\frac{f(\tau)}{f(\tau_1)}\left\{\sum_0^\infty A_n(tw)^n + \left[f^{(1)}(\tau) - f^{(1)}(\tau_1)\right]\sum_2^\infty \frac{A_n}{n}(tw)^n\right\} \quad |w| < 1$$

But it is easily seen that

$$-\sum_2^\infty \frac{A_n}{n}(tw)^n = \int_0^{tw}\left[W_o(w) + A_o\right]\frac{dw}{w}$$

Therefore, $W(w,\tau)$ can be represented by two closed functions

$$W(w,\tau) = \frac{f(\tau)}{f(\tau_1)}\left\{W_o(wt) + \left[f^{(1)}(\tau) - f^{(1)}(\tau_1)\right]\int_0^{tw}\left[W_o(w) + A_o\right]\frac{dw}{w}\right\} \qquad (222)$$

The function represented by the integral, when continued, will give rise to a term $\log_e \tau w$. No longer is $w = 0$ a regular point; the singularity must be either at $w = 1$ or at origin.

APPENDIX C

ASYMPTOTIC REPRESENTATION OF HYPERGEOMETRIC FUNCTIONS

The asymptotic integration of the hypergeometric equation for a large positive parameter ν has been discussed (reference 8). It was shown that its solutions under this condition are of the exponential type when the variable ν is in the interval $\delta \leq \tau \leq \frac{1}{2\beta + 1} - \delta$, where $\delta > 0$, and of the oscillatory type when τ is in the interval $\frac{1}{2\beta + 1} + \delta \leq \tau \leq 1 - \delta$. The point $\tau = \frac{1}{2\beta + 1}$ is a singularity. Thus, to the first approximation, the hypergeometric functions for the interval are

$$F_{\nu}(\tau) \sim f(\tau)T^{\nu}(\tau)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{223}$$

$$F_{-\nu}(\tau) \sim f(\tau)T^{-\nu}(\tau)\left[1 + O\left(\frac{1}{\nu}\right)\right] \qquad \nu > N \tag{224}$$

where

$$\left.\begin{aligned} f(\tau) &= (1 - \tau)^{\alpha^2/4}(1 - \alpha^2\tau)^{-1/4} \\ T(\tau) &= \frac{2}{(1 + \alpha)^{\alpha}}\frac{\left[\alpha(1 - \tau)^{1/2} + (1 - \alpha^2\tau)^{1/2}\right]^{\alpha}}{(1 - \tau)^{1/2} + (1 - \alpha^2\tau)^{1/2}} \end{aligned}\right\} \tag{225}$$

Here $O\left(\frac{1}{\nu}\right)$ in each case denotes the fact that the term is uniformly of the order of $1/\nu$ when ν is sufficiently large and is a function of $1/\nu$. For the interval $\frac{1}{2\beta + 1} < \tau < 1$, the hypergeometric functions are

$$F_{\nu}(\tau) \sim f(\tau)T^{\nu}(\tau)\cos\left(\nu\omega - \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{226}$$

$$F_{-\nu}(\tau) \sim \frac{1}{2}f(\tau)T^{-\nu}(\tau)\cos\left(\nu\omega + \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{227}$$

where

$$\left.\begin{aligned} f(\tau) &= 2(1-\tau)^{\alpha^2/4}(\alpha^2\tau - 1)^{-1/4} \\ T(\tau) &= 2\frac{(2\beta)^{\alpha/2}}{(1+\alpha)^{\alpha}}\frac{1}{\sqrt{2\beta\tau}} \end{aligned}\right\} \tag{228}$$

$$\omega(\tau) = \alpha \tan^{-1}\frac{1}{\alpha}\sqrt{\frac{\alpha^2\tau - 1}{1-\tau}} - \tan^{-1}\sqrt{\frac{\alpha^2\tau - 1}{1-\tau}} \tag{229}$$

The values of $f(\tau)$, $T(\tau)$, and $\omega(\tau)$ are given in table 1. Here the parameter ν may be any positive large number. When it is a large positive integer, these formulas will automatically represent the hypergeometric functions defined by expression (16).

In the respective domains of validity, the asymptotic expansions may be differentiated with respect to τ. To the same order of approximation, it can be shown that for $\nu > N$ in the interval

$$F_{\nu,1}(\tau) \sim g(\tau)T^{\nu}(\tau)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{230}$$

$$F_{-\nu,1}(\tau) \sim g(\tau)T^{-\nu}(\tau)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{231}$$

where

$$\left.\begin{aligned} g(\tau) &= \frac{2f(\tau)}{(1-\tau)\left[1 + \xi_o(\tau)\right]} \\ \xi_o(\tau) &= \sqrt{\frac{1-\alpha^2\tau}{1-\tau}} \end{aligned}\right\} \tag{232}$$

and in the interval $\frac{1}{2\beta + 1} < \tau < 1$

$$F_{\nu,1}(\tau) \sim g(\tau)T^{\nu}(\tau) \cos\left(\nu\omega - \mu^o - \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{233}$$

$$F_{-\nu,1}(\tau) \sim \frac{1}{2}g(\tau)T^{-\nu}(\tau) \cos\left(\nu\omega + \mu^o + \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{234}$$

where

$$\left.\begin{aligned} q(\tau) &= \frac{2f(\tau)}{\sqrt{2\beta\tau(1-\tau)}} \\ \mu^{o}(\tau) &= \cos^{-1}\sqrt{\frac{1-\tau}{2\beta\tau}} \end{aligned}\right\} \tag{235}$$

Here $F_{\nu,1}(\tau)$ and $F_{-\nu,1}(\tau)$ represent respectively the functions $F(a_\nu + 1, b_\nu + 1; c_\nu + 1; \tau)$ and $F(1 + a_\nu - c_\nu + 1, 1 + b_\nu - c_\nu + 1; 2 - c_\nu + 1; \tau)$. When ν takes integral values n, expressions (231) and (234) will represent asymptotically $F_{-n,1}(\tau)$. The values of $g(\tau)$ and $\mu^{o}(\tau)$ are given in table 1.

In practice, a more accurate representation for the functions $F_\nu(\tau)$ and $F_{-\nu}(\tau)$ and their derivatives is often required, especially in the interval $0 \leq \tau < \frac{1}{2\beta + 1}$. The fact that this is the case is quite evident in part III. In order to carry the approximation to the second order, according to the method given in reference 8, a simple evaluation gives

$$F_\nu(\tau) \sim f(\tau)T^{\nu}(\tau)\left[1 + \frac{f^{(1)}(\tau)}{\nu} + O\left(\frac{1}{\nu^2}\right)\right] \tag{236}$$

$$0 \leq \tau < \frac{1}{2\beta + 1}$$

$$F_{-\nu}(\tau) \sim f(\tau)T^{-\nu}(\tau)\left[1 - \frac{f^{(1)}(\tau)}{\nu} + O\left(\frac{1}{\nu^2}\right)\right] \tag{237}$$

where

$$f^{(1)}(\tau) = \frac{1}{16}\left[\frac{\alpha^2(\alpha^2 + 2)}{\beta}(1 - \xi_o) - \frac{8\beta^2}{\alpha}\log_e\frac{(1-\tau)^{1/2}(\alpha - \xi_o)}{\alpha - 1}\right.$$

$$\left. + \frac{6\beta + 1}{3\beta}\left(\xi_o^{-1} - 1\right) + \frac{10}{3}\left(1 - \xi_o^{-3}\right)\right] \tag{238}$$

Similarly, by differentiation with respect to τ of expressions (236) and (237), the following expressions are obtained:

$$F_{\nu,1}(\tau) \sim g(\tau)T^{\nu}(\tau)\left[1 + \frac{g^{(1)}(\tau)}{\nu} + O\left(\frac{1}{\nu}\right)\right] \tag{239}$$

$$0 \le \tau < \frac{1}{2\beta + 1}$$

$$F_{-\nu,1}(\tau) \sim g(\tau)T^{-\nu}(\tau)\left[1 - \frac{g^{(1)}(\tau)}{\nu} + O\left(\frac{1}{\nu}\right)\right] \tag{240}$$

where

$$g^{(1)}(\tau) = f^{(1)}(\tau) + \frac{\alpha^2}{4\beta}(1 + \xi_0)(1 - \xi_0^{-2}) \tag{241}$$

By comparison of the values of $f^{(1)}(\tau)$ and $g^{(1)}(\tau)$, the first approximation of $F_{\nu}(\tau)$ and $F_{-\nu}(\tau)$ is seen to be superior to that of $F_{\nu,1}(\tau)$ and $F_{-\nu,1}(\tau)$. The values $f^{(1)}(\tau)$ and $g^{(1)}(\tau)$ are given in table 1.

For $\gamma = 1.405$ and $\nu = n + 1/2$ and $\nu = n$, n being a positive integer, the two groups of functions $F_{\nu}(\tau)$, $F_{-\nu}(\tau)$ and $F_{\nu,1}(\tau)$, $F_{-\nu,1}(\tau)$ with their asymptotic expressions were calculated for values of τ varying from 0 to 0.34 and for values of n from 1 to 10. The results are presented in table 2.

APPENDIX D

ASYMPTOTIC REPRESENTATION FOR $\xi_\nu(\tau)$ AND $\xi_{-\nu}(\tau)$

Next in importance are the functions $\xi_\nu(\tau)$ and $\xi_{-\nu}(\tau)$, defined by equation (29). They are associated with the particular solutions of $\varphi(q,\theta)$, and eventually, through the latter, will appear in the various functions in the problem of compressible flow. As was shown in part II, the whole scheme is based on the stream function $\psi(q,\theta)$, and its determination depends on the efficiency of the determination of the coefficients of the power series representing $\psi(q,\theta)$. In order to facilitate such evaluation, the asymptotic expressions are again powerful tools. In deriving these expansions, it is also convenient to start from the differential equation for $\xi_\nu(\tau)$ which, from equation (13), is as follows:

$$\xi_\nu'(\tau) + \frac{\beta}{1-\tau}\xi_\nu(\tau) + \frac{\nu}{2\tau}\left[\xi_\nu^{\,2}(\tau) - \xi_o^{\,2}\right] = 0 \tag{242}$$

This is the celebrated Riccati equation. This proves to be an adequate form for the asymptotic development of $\xi_\nu(\tau)$ in the interval $0 \leq \tau < \frac{1}{2\beta + 1}$. Suppose the expansions are of the following forms:

$$\xi_\nu(\tau) \sim \xi_o + \frac{\xi^{(1)}}{\nu} + \frac{\xi^{(2)}}{\nu^2} + O\left(\frac{1}{\nu^3}\right) \tag{243}$$

$$\xi_{-\nu}(\tau) \sim -\xi_o + \frac{\xi^{(1)}}{\nu} - \frac{\xi^{(2)}}{\nu^2} + O\left(\frac{1}{\nu^3}\right) \tag{244}$$

where $\xi_o(\tau)$ is defined by equation (232). Substituting these expressions in equation (242) and equating the coefficients of various values of ν^{-s} to zero yields for the interval

$$\xi^{(1)}(\tau) = \frac{\alpha^2\tau}{2(1-\tau)}\left(\xi_o^{-2} - 1\right) \tag{245}$$

$$\xi^{(2)}(\tau) = \frac{\alpha^2\tau(\beta\tau + 2)}{4\xi_o(1-\tau)^2}\left(1 - \xi_o^{-4}\right) - \frac{\xi^{(1)}(\tau)}{2\alpha^2\xi_o} \tag{246}$$

The values of $\xi^{(1)}$ and $\xi^{(2)}$ are given in table 1.

APPENDIX E

PROOF OF THEOREM 2

The first part of the proof, namely, the series

$$\psi(q,\theta) = \sum_{2}^{\infty} \tilde{A}_n q^n F_n^{(r)}(\tau) \sin n\theta \qquad q < 1$$

which is absolutely and uniformly convergent in any closed domain in $q < 1$, can be similarly carried out (reference 1), since according to equation (159) it can be deduced that for large values of n

$$\left|\tilde{A}_n\right| < M\left|A_n\right|$$

where M is a constant independent of n. Therefore, it follows that for $\tau_1 << \frac{1}{2\beta + 1}$ and $n > N$

$$\left|\tilde{A}_n q^n F_n^{(r)}(\tau)\right| < M\left|A_n(tq)^n\right|$$

For the series (equation (51)) it is noted that for $\tau \lessgtr \frac{1}{2\beta + 1}$ and $\nu > N$

$$\left|F_\nu^{(r)}(\tau)\right| < Mt^\nu$$

$$\left|F_{-\nu}^{(r)}(\tau)\right| < Mt^{-\nu}$$

where the region $\frac{1}{2\beta + 1} - \delta \leq \leq \frac{1}{2\beta + 1} + \delta$, where $\delta > 0$ is assumed to be excluded. Here all the M's are different but independent of n. Furthermore, from equation (160), $\tilde{D}_n$ is bounded by

$$\left|\tilde{D}_n\right| < \frac{M}{n}$$

Consequently, for $n > N$, the following inequalities are obtained:

$$\left|B_n q^{\nu} F_{\nu}^{(r)}(\tau)\right| < M \left|B_n (qt)^{\nu}\right| \qquad qt < V$$

$$\left|C_n q^{-\nu} F_{-\nu}^{(r)}(\tau)\right| < M \left|C_n (qt)^{-\nu}\right| \qquad qt > 1$$

$$\left|\tilde{D}_n q^{-n} F_{-n}^{(r)}(\tau)\right| < M \frac{(tq)^{-n}}{n}$$

This completes the proof.

APPENDIX F

DEDUCTION OF IDENTITIES (153), (154), AND (155)

Consider the stream function of the similar incompressible flow

$$\psi_o(q,\theta) = \sum_2^{\infty} A_n q^n \sin n\theta \qquad q < 1$$

$$\psi_o(q,\theta) = \sum_0^{\infty} \left(B_n q^{\nu} + C_n q^{-\nu}\right) \cos \nu\theta \qquad 1 < q < V$$

At the circle of convergence the conditions of continuity give

$$\sum_2^{\infty} A_n \sin n\theta = \sum_0^{\infty} (B_n + C_n) \cos \nu\theta$$

$$\sum_2^{\infty} nA_n \sin n\theta = \sum_0^{\infty} \nu(B_n - C_n) \cos \nu\theta$$

Since the limits exist by hypothesis, by multiplying both sides by $\sin n\theta$, term-by-term integration then yields

$$A_n = \frac{1}{\pi} \sum_0^{\infty} (B_m + C_m) I_{n\mu} \tag{247}$$

$$nA_n = \frac{1}{\pi} \sum_0^{\infty} \mu(B_m - C_m) I_{n\mu} \tag{248}$$

where

$$I_{n\mu} = \frac{1}{n + \mu} + \frac{1}{n - \mu}$$

$$\mu = m + \frac{1}{2}$$

By use of appendix B, from

$$-\sum_{2}^{\infty}\frac{A_n}{n}w^n = \int_0^w\left[W_o(w) + A_o\right]\frac{dw}{w}$$

the following relation by continuation can be deduced. Denote this function by $\Lambda(w)$. Then

$$\Lambda(w) = -\sum_{2}^{\infty}\frac{A_n}{n}w^n \qquad |w| < 1$$

$$\Lambda(w) = i\sum_{0}^{\infty}\frac{1}{\nu}\left(B_n w^{\nu} - C_n w^{-\nu}\right) - A_o \log_e w + \text{Constant} \qquad 1 < |w| < V$$

where the constant of integration can be determined if the form of $\Lambda(w)$ is known. At the circle of convergence it can similarly be shown, by considering the imaginary part of $\Lambda(w)$, that

$$\frac{A_n}{n} = \frac{1}{\pi}\sum_{0}^{\infty}\frac{1}{\mu}(B_m - C_m)I_{n\mu} + \frac{4}{n} \tag{249}$$

where the constant term is eliminated by integration.

Moreover, if $\Lambda(w) = \sigma + i\chi$, it follows that

$$\sigma(q,\theta) = -\sum_{2}^{\infty}\frac{A_n}{n}q^n \cos n\theta \tag{250}$$

$$q < 1$$

$$\chi(q,\theta) = \sum_{2}^{\infty}\frac{A_n}{n}q^n \sin n\theta \tag{251}$$

$$\sigma(q,\theta) = \sum_{0}^{\infty}\frac{1}{\nu}\left(B_n q^{\nu} + C_n q^{-\nu}\right)\sin \nu\theta + 2\log_e q + \sigma_o \tag{252}$$

$$1 < q < V$$

$$\chi(q,\theta) = \sum_{0}^{\infty}\frac{1}{\nu}\left(B_n q^{\nu} - C_n q^{-\nu}\right)\cos \nu\theta - A_o(\pi - \theta) \tag{253}$$

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TABLE 1

τ	f	g	T	$(1-\tau)^{-\beta}$	M	$-\int^{T}(1-\tau)^{\beta}\frac{d\tau}{\tau}$	$-f^{(1)}$	$-g^{(1)}$	ξ_o	$\xi^{(1)}$	$\xi^{(2)}$
0.01	1.00038	1.02341	0.987591	1.02513	0.22334	4.62978	0.0070935	0.069428	0.974740	0.001575	0.003390
.02	1.00162	1.04920	.975050	1.05115	.31746	3.961051	.0035781	.134863	.948272	.006791	.012038
.03	1.00384	1.07775	.962369	1.07811	.39081	3.57983	.0089751	.217119	.920473	.016553	.041943
.04	1.00722	1.10954	.94935	1.10605	.45361	3.31621	.0179196	.312501	.891201	.032050	.088904
.05	1.01199	1.14526	.936535	1.13502	.50981	3.11694	.0317171	.424516	.860285	.054595	.166883
.06	1.01844	1.18570	.923360	1.16507	.56144	2.95832	.0522626	.558038	.827521	.087236	.297127
.07	1.02678	1.23176	.909988	1.21197	.60967	2.82770	.0823873	.720024	.792655	.132211	.506825
.08	1.03804	1.28554	.896397	1.22861	.65530	2.71751	.1264739	.920767	.755371	.194307	.849851
.09	1.05245	1.34851	.882567	1.26221	.69886	2.62291	.1915979	1.175123	.715263	.280335	1.423202
.10	1.07125	1.42397	.868464	1.29712	.74074	2.54053	.2898099	1.511895	.671791	.401099	2.416900
.11	1.09610	1.51651	.854048	1.33341	.781252	2.46804	.4430778	1.972760	.624221	.574823	4.235747
.12	1.12968	1.63378	.839267	1.37113	.82061	2.40366	.6952314	2.643391	.571489	.834809	7.848095
.13	1.17684	1.78931	.824049	1.41038	.859012	2.34610	1.145421	3.704775	.511954	1.24908	11.823761
.14	1.24743	2.01063	.808289	1.45122	.89661	2.29428	2.064247	5.620614	.442827	1.981513	37.881541
.15	1.36648	2.36673	.791805	1.49374	.93352	2.24741	4.465971	10.003688	.358525	3.55231	125.02174
.16	1.63348	3.12791	.774259	1.53803	.96986	2.20482	16.03668	27.88238	.243674	8.95921	962.9315

τ	f	g	T	$(1-\tau)^{-\beta}$	M	$-\int^{T}(1-\tau)^{\beta}\frac{d\tau}{\tau}$	ω^{o}	μ^{o}
0.17	4.85733	11.63759	0.754293	1.58419	1.0057	2.16598	0.01806	6.10861
.18	2.90766	6.81071	.733041	1.63232	1.0412	2.13040	.36479	16.16389
.19	2.44420	5.60734	.713490	1.68253	1.0763	2.09780	.89674	21.69972
.20	2.18183	4.90915	.695424	1.73494	1.1111	2.06776	1.55446	25.84194
.21	1.99922	4.41764	.678664	1.78964	1.1457	2.04007	2.28050	29.21389
.22	1.85891	4.03871	.663061	1.84686	1.1802	2.01448	3.06999	32.07833
.23	1.74464	3.73114	.648486	1.90665	1.2145	1.99078	3.90948	34.57667
.24	1.64795	3.47275	.634832	1.96919	1.2498	1.96882	4.78568	36.79528
.25	1.56391	3.25054	.622006	2.03465	1.2830	1.94842	5.69266	38.79222
.26	1.48938	3.05592	.609927	2.10322	1.3172	1.92945	6.62614	40.60833
.27	1.42228	2.88326	.598526	2.17508	1.3515	1.91181	7.58100	42.27445
.28	1.36113	2.72830	.587740	2.25043	1.3858	1.89537	8.55467	43.81278
.29	1.30486	2.58811	.577518	2.32950	1.4202	1.88004	9.54119	45.24417
.30	1.25267	2.46020	.567811	2.41254	1.4548	1.86573	10.54604	46.57667
.32	1.15821	2.23459	.549781	2.59154	1.5244	1.83991	12.58533	49.00611
.34	1.07420	2.04082	.533366	2.78978	1.5950	1.81733	14.66453	51.17333
.36	.998334	1.87187	.518338	3.01001	1.6667	1.79760	16.77201	53.13000
.38	.929024	1.72360	.504513	3.25546	1.7398	1.78031	18.90685	54.91444
.40	.865127	1.58933	.491739	3.53000	1.8140	1.76517	21.06373	56.55500

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION

$F_\nu(\tau)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.04	9.7568 ,-1	9.2786 ,-1	8.8182 ,-1	8.3781 ,-1	7.9586 ,-1	7.5593 ,-1	7.1796 ,-1	6.8186 ,-1	6.4755 ,-1	6.1495 ,-1	5.8397 ,-1
.06	9.6380 ,-1	8.9322 ,-1	8.2658 ,-1	7.6437 ,-1	7.0654 ,-1	6.5292 ,-1	6.0312 ,-1	5.5732 ,-1	5.1482 ,-1	4.7553 ,-1	4.3922 ,-1
.08	9.5211 ,-1	8.5952 ,-1	7.7384 ,-1	6.9574 ,-1	6.2504 ,-1	5.6122 ,-1	5.0375 ,-1	4.5205 ,-1	4.0559 ,-1	3.6385 ,-1	3.2636 ,-1
.10	9.4059 ,-1	8.2675 ,-1	7.2351 ,-1	6.3173 ,-1	5.5084 ,-1	4.7989 ,-1	4.1782 ,-1	3.6361 ,-1	3.1634 ,-1	2.7514 ,-1	2.3925 ,-1
.12	9.2926 ,-1	7.9489 ,-1	6.7554 ,-1	5.7211 ,-1	4.8349 ,-1	4.0802 ,-1	3.4398 ,-1	2.8978 ,-1	2.4398 ,-1	2.0533 ,-1	1.7274 ,-1
.14	9.1811 ,-1	7.6395 ,-1	6.2986 ,-1	5.1669 ,-1	4.2251 ,-1	3.4476 ,-1	2.8088 ,-1	2.2856 ,-1	1.8582 ,-1	1.5097 ,-1	1.2258 ,-1
.16	9.0714 ,-1	7.3389 ,-1	5.8641 ,-1	4.6528 ,-1	3.6750 ,-1	2.8934 ,-1	2.2727 ,-1	1.7820 ,-1	1.3952 ,-1	1.0911 ,-1	8.5244 ,-2
.18	8.9636 ,-1	7.0472 ,-1	5.4513 ,-1	4.1768 ,-1	3.1800 ,-1	2.4101 ,-1	1.8203 ,-1	1.3710 ,-1	1.0304 ,-1	7.7297 ,-2	5.7890 ,-2
.20	8.8574 ,-1	6.7642 ,-1	5.0595 ,-1	3.7371 ,-1	2.7364 ,-1	1.9908 ,-1	1.4411 ,-1	1.0389 ,-1	7.4637 ,-2	5.3469 ,-2	3.8199 ,-2
.22	8.7531 ,-1	6.4897 ,-1	4.6882 ,-1	3.3317 ,-1	2.3403 ,-1	1.6291 ,-1	1.1258 ,-1	7.7322 ,-2	5.2823 ,-2	3.5922 ,-2	2.4313 ,-2
.24	8.6505 ,-1	6.2238 ,-1	4.3366 ,-1	2.9590 ,-1	1.9879 ,-1	1.3189 ,-1	8.6594 ,-2	5.6325 ,02	3.6325 ,-2	2.3252 ,-2	1.4759 ,-2
.26	8.5497 ,-1	5.9662 ,-1	4.0043 ,-1	2.6172 ,-1	1.6759 ,-1	1.0548 ,-1	6.5384 ,-2	3.9952 ,-2	2.4076 ,-2	1.4319 ,-2	8.3804 ,-2
.28	8.4507 ,-1	5.7168 ,-1	3.6905 ,-1	2.3046 ,-1	1.4009 ,-1	8.3160 ,-1	4.8264 ,-2	2.7386 ,-2	1.5175 ,-2	8.2015 ,-3	4.2820 ,-3
.30	8.3533 ,-1	5.4756 ,-1	3.3947 ,-1	2.0195 ,-1	1.1599 ,-1	6.4449 ,-2	3.4623 ,-2	1.7922 ,-2	8.8760 ,-3	4.1658 ,-3	1.7807 ,-3
.32	8.2577 ,-1	5.2423 ,-1	3.1162 ,-1	1.7604 ,-1	9.4975 ,-2	4.8914 ,-2	2.3915 ,-2	1.0955 ,-2	4.5695 ,-3	1.6349 ,-3	3.6396 ,-4
.34	8.1638 ,-1	5.0169 ,-1	2.8545 ,-1	1.5256 ,-1	7.6774 ,-2	3.6160 ,-2	1.5659 ,-2	5.9706 ,-3	1.7564 ,-3	1.6177 ,-4	-3.4465 ,-4

$F_{-\nu}(\tau)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.04	1.0242 ,0	1.0630 ,0	1.1548 ,0	1.2209 ,0	1.2833 ,0	1.3485 ,0	1.4177 ,0	1.4921 ,0	1.5694 ,0	1.6519 ,0	1.7390 ,0
.06	1.0358 ,0	1.0868 ,0	1.2444 ,0	1.3712 ,0	1.4892 ,0	1.6098 ,0	1.7382 ,0	1.8775 ,0	2.0263 ,0	2.1929 ,0	2.3989 ,0
.08	1.0472 ,0	1.1059 ,0	1.3338 ,0	1.5428 ,0	1.7468 ,0	1.9578 ,0	2.1838 ,0	2.4308 ,0	2.7038 ,0	3.0071 ,0	3.3452 ,0
.10	1.0584 ,0	1.1206 ,0	1.4177 ,0	1.7261 ,0	2.0512 ,0	2.4037 ,0	2.7935 ,0	3.2302 ,0	3.7237 ,0	4.2847 ,0	4.9249 ,0
.12	1.0693 ,0	1.1312 ,0	1.4921 ,0	1.9088 ,0	2.3860 ,0	2.9384 ,0	3.5830 ,0	4.3394 ,0	5.2299 ,0	6.2810 ,0	7.5235 ,0
.14	1.0799 ,0	1.1381 ,0	1.5537 ,0	2.0777 ,0	2.7246 ,0	3.5256 ,0	4.5196 ,0	5.7548 ,0	7.2900 ,0	9.1979 ,0	1.1569 ,1
.16	1.0903 ,0	1.1416 ,0	1.6000 ,0	2.2192 ,0	3.0326 ,0	4.1006 ,0	5.5030 ,0	7.3443 ,0	9.7604 ,0	1.2929 ,1	1.7080 ,1
.18	1.1005 ,0	1.1420 ,0	1.6291 ,0	2.3209 ,0	3.2712 ,0	4.5737 ,0	6.3578 ,0	8.8005 ,0	1.2143 ,1	1.6712 ,1	2.2954 ,1
.20	1.1104 ,0	1.1395 ,0	1.6400 ,0	2.3717 ,0	3.4001 ,0	4.8378 ,0	6.8420 ,0	9.6228 ,0	1.3482 ,1	1.8831 ,1	2.6186 ,1
.22	1.1201 ,0	1.1344 ,0	1.6321 ,0	2.3629 ,0	3.3819 ,0	4.7805 ,0	6.6716 ,0	9.1581 ,0	1.2445 ,1	1.6550 ,1	2.1493 ,1
.24	1.1295 ,0	1.1270 ,0	1.6054 ,0	2.2882 ,0	3.1849 ,0	4.2963 ,0	5.5580 ,0	6.7611 ,0	7.4100 ,0	6.4660 ,0	1.9037 ,0
.26	1.1387 ,0	1.1174 ,0	1.5602 ,0	2.1437 ,0	2.7862 ,0	3.3015 ,0	3.2562 ,0	1.7335 ,0	-3.0847 ,0	-1.4652 ,1	-3.9320 ,1
.28	1.1477 ,0	1.1061 ,0	1.4974 ,0	1.9288 ,0	2.1737 ,0	1.7457 ,0	-3.8653 ,-1	-6.2885 ,0	-1.9917 ,1	-4.8592 ,1	-1.0547 ,2
.30	1.1565 ,0	1.0930 ,0	1.4181 ,0	1.6454 ,0	1.3477 ,0	-3.7756 ,-1	-5.3857 ,0	-1.7297 ,1	-4.2875 ,1	-9.4278 ,1	-1.9258 ,2
.32	1.1650 ,0	1.0786 ,0	1.3238 ,0	1.2978 ,0	3.2159 ,-1	-3.0244 ,0	-1.1587 ,1	-3.0762 ,1	-7.0252 ,1	-1.4660 ,2	-2.8631 ,2
.34	1.1733 ,0	1.0630 ,0	1.2161 ,0	8.9307 ,-1	-8.7844 ,-1	-6.0980 ,0	-1.8655 ,1	-4.6571 ,1	-9.8661 ,1	-1.9592 ,2	-3.6305 ,2

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TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$F_{\nu,1}(\tau)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.04	9.6983 ,-1	9.4805 ,-1	9.1526 ,-1	8.7850 ,-1	8.4053 ,-1	8.0264 ,-1	7.6547 ,-1	7.2938 ,-1	6.9454 ,-1	6.6106 ,-1	6.2895 ,-1
.05	9.5485 ,-1	9.2252 ,-1	8.7448 ,-1	8.2154 ,-1	7.6796 ,-1	7.1568 ,-1	6.6559 ,-1	6.1814 ,-1	5.7346 ,-1	5.3161 ,-1	4.9251 ,-1
.08	8.9730 ,-1	8.9730 ,-1	8.3474 ,-1	7.6703 ,-1	6.9739 ,-1	6.3166 ,-1	5.2090 ,-1	5.2090 ,-1	4.7024 ,-1	4.2404 ,-1	3.8203 ,-1
.10	9.2515 ,-1	8.7239 ,-1	7.9604 ,-1	7.1492 ,-1	6.3626 ,-1	5.6302 ,-1	4.9629 ,-1	4.3627 ,-1	3.8272 ,-1	3.3522 ,-1	2.9327 ,-1
.12	9.1041 ,-1	8.4779 ,-1	7.5838 ,-1	6.6516 ,-1	5.7678 ,-1	4.9650 ,-1	4.2527 ,-1	3.6296 ,-1	3.0984 ,-1	2.6242 ,-1	2.2254 ,-1
.14	8.9575 ,-1	8.2349 ,-1	7.2175 ,-1	6.1770 ,-1	5.2131 ,-1	4.3599 ,-1	3.6235 ,-1	2.9977 ,-1	2.4714 ,-1	2.0320 ,-1	1.6672 ,-1
.16	8.8118 ,-1	7.9951 ,-1	6.8614 ,-1	5.7249 ,-1	4.6969 ,-1	3.8111 ,-1	3.0683 ,-1	2.4561 ,-1	1.9573 ,-1	1.5544 ,-1	1.2310 ,-1
.18	8.6668 ,-1	7.7585 ,-1	6.5154 ,-1	5.2947 ,-1	4.2176 ,-1	3.3149 ,-1	2.5806 ,-1	1.9945 ,-1	1.5329 ,-1	1.1727 ,-1	8.9399 ,-2
.20	8.5227 ,-1	7.5249 ,-1	6.1796 ,-1	4.8860 ,-1	3.7735 ,-1	2.8679 ,-1	2.1542 ,-1	1.6037 ,-1	1.1853 ,-1	8.7097 ,-2	6.3695 ,-2
.22	8.3794 ,-1	7.2946 ,-1	5.8539 ,-1	4.4982 ,-1	3.3630 ,-1	2.4666 ,-1	1.7835 ,-1	1.2752 ,-1	9.0343 ,-2	6.3512 ,-2	4.4368 ,-2
.24	8.2369 ,-1	7.0675 ,-1	5.5380 ,-1	4.1307 ,-1	2.9845 ,-1	2.1078 ,-1	1.4629 ,-1	1.0011 ,-1	6.7708 ,-2	4.5320 ,-2	3.0075 ,-2
.26	8.0953 ,-1	6.8436 ,-1	5.2321 ,-1	3.7832 ,-1	2.6365 ,-1	1.7883 ,-1	1.1874 ,-1	7.7449 ,-2	4.9744 ,-2	3.1499 ,-2	1.9704 ,-2
.28	7.9545 ,-1	6.6229 ,-1	4.9361 ,-1	3.4550 ,-1	2.3175 ,-1	1.5053 ,-1	9.5231 ,-2	5.8893 ,-2	3.5675 ,-2	2.1181 ,-2	1.2349 ,-2
.30	7.8147 ,-1	6.4055 ,-1	4.6498 ,-1	3.1457 ,-1	2.0260 ,-1	1.2557 ,-1	7.5324 ,-2	4.3864 ,-2	2.4825 ,-2	1.3637 ,-2	7.2747 ,-3
.32	7.6757 ,-1	6.1914 ,-1	4.3732 ,-1	2.8546 ,-1	1.7606 ,-1	1.0369 ,-1	5.8610 ,-2	3.1845 ,-2	1.6607 ,-2	8.2582 ,-3	3.8945 ,-3
.34	7.5376 ,-1	5.9806 ,-1	4.1062 ,-1	2.5813 ,-1	1.5197 ,-1	8.4623 ,-1	4.4715 ,-2	2.2374 ,-2	1.0517 ,-2	4.5429 ,-3	1.7440 ,-3

$F_{-\nu,1}(\tau)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.04	9.5633 ,-1	7.0820 ,-1	1.4234 ,0	1.5898 ,0	1.6278 ,0	1.6554 ,0	1.6968 ,0	1.7689 ,0	1.8234 ,0	1.9032	1.9911 ,0
.06	9.3487 ,-1	5.7770 ,-1	1.4649 ,0	1.8778 ,0	2.0849 ,0	2.2203 ,0	2.3355 ,0	2.4731 ,0	2.5910 ,0	2.7851	3.3138 ,0
.08	9.1375 ,-1	4.5436 ,-1	1.4160 ,0	2.0750 ,0	2.5526 ,0	2.9348 ,0	3.2537 ,0	3.5806 ,0	3.9223 ,0	4.2932 ,0	4.7036 ,0
.10	8.9296 ,-1	3.4085 ,-1	1.2926 ,0	2.1429 ,0	2.9088 ,0	3.6372 ,0	4.3633 ,0	5.1151 ,0	5.9166 ,0	6.7896 ,0	7.7547 ,0
.12	8.7237 ,-1	2.3599 ,-1	1.1092 ,-1	2.0605 ,0	3.0767 ,0	4.1927 ,0	5.4438 ,0	6.8671 ,0	8.5020 ,0	1.0394 ,1	1.2592 ,1
.14	8.5214 ,-1	1.3939 ,-1	8.7917 ,-1	1.8210 ,0	2.9660 ,0	4.3755 ,0	6.1244 ,0	8.3050 ,0	1.1027 ,1	1.4428 ,1	1.8672 ,1
.16	8.3215 ,-1	5.0743 ,-2	6.1465 ,-1	1.4299 ,0	2.5201 ,0	3.9821 ,0	5.9467 ,0	8.5864 ,0	1.2126 ,1	1.6866 ,1	2.3188 ,1
.18	8.1253 ,-1	3.0225 ,-2	3.2667 ,-1	9.0162 ,-1	1.7128 ,-	2.8523 ,0	4.4517 ,0	6.6921 ,0	9.8209 ,0	1.4183 ,1	2.0225 ,1
.20	7.9316 ,-1	-1.0381 ,-1	2.5108 ,-2	2.5808 ,-1	5.5107 ,-1	8.9845 ,-1	1.2834 ,0	1.6485 ,0	1.9131 ,0	1.9326 ,0	1.2782 ,0
.22	7.7409 ,-1	-1.7032 ,-1	-2.8113 ,-1	-4.7390 ,-1	-9.2694 ,-1	-1.8743 ,0	-3.7128 ,0	-7.1668 ,0	-1.3205 ,1	-2.3824 ,1	-4.2083 ,1
.24	7.5530 ,-1	-2.3004 ,-1	-5.8418 ,-1	-1.2638 ,0	-2.6540 ,0	-5.3605 ,0	-1.0424 ,1	-1.9627 ,1	-3.5985 ,1	-6.4489 ,1	-1.1358 ,2
.26	7.3683 ,-1	-2.8327 ,-1	-8.7723 ,-1	-2.0799 ,0	-4.5416 ,0	-9.3549 ,0	-1.8424 ,1	-3.5036 ,1	-6.4761 ,1	-1.1686 ,2	-2.0680 ,2
.28	7.1867 ,-1	-3.3027 ,-1	-1.1543 ,0	-2.8898 ,0	-6.4835 ,0	-1.3568 ,1	-2.6990 ,1	-5.1607 ,1	-9.5502 ,1	-1.7181 ,2	-3.0115 ,2
.30	7.0078 ,-1	-3.7136 ,-1	-1.4104 ,0	-3.6621 ,0	-8.3641 ,0	-1.7649 ,1	-3.5149 ,1	-6.6813 ,1	-1.2193 ,2	-2.1458 ,2	-3.6171 ,2
.32	6.8320 ,-1	-4.0682 ,-1	-1.6413 ,0	-4.3680 ,0	-1.0056 ,1	-2.1213 ,1	-4.1774 ,1	-7.7603 ,1	-1.3625 ,2	-2.2802 ,2	-3.4477 ,2
.34	6.6592 ,-1	-4.3695 ,-1	-1.8438 ,0	-4.9811 ,0	-1.1477 ,1	-2.3873 ,1	-4.5708 ,1	-7.9730 ,1	-1.3050 ,2	-2.4367 ,2	-2.2428 ,2

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$F_n(\tau)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	9.7556 ,-1	9.5146 ,-1	9.2787 ,-1	9.0482 ,01	8.8232 ,-1	8.6035 ,-1	8.3893 ,-1	8.1803 ,-1	7.9764 ,-1	7.7776 ,-1
.04	9.5159 ,-1	9.0459 ,-1	8.5955 ,-1	8.1658 ,-1	7.7565 ,-1	7.3671 ,-1	6.9968 ,-1	6.6449 ,-1	6.3104 ,-1	5.9927 ,-1
.06	9.2808 ,-1	8.5936 ,-1	7.9492 ,-1	7.3491 ,-1	6.7922 ,-1	6.2761 ,-1	5.7984 ,-1	5.3565 ,-1	4.9479 ,-1	4.5702 ,-1
.08	9.0507 ,-1	8.1574 ,-1	7.3384 ,-1	6.5949 ,-1	5.9230 ,-1	5.3173 ,-1	4.7722 ,-1	4.2820 ,-1	3.8416 ,-1	3.4460 ,-1
.10	8.8252 ,-1	7.7370 ,-1	6.7620 ,-1	5.8998 ,-1	5.1419 ,-1	4.4781 ,-1	3.8979 ,-1	3.3917 ,-1	2.9503 ,-1	2.5658 ,-1
.12	8.6044 ,-1	7.3320 ,-1	6.2188 ,-1	5.2605 ,-1	4.4422 ,-1	3.7467 ,-1	3.1574 ,-1	2.6591 ,-1	2.2383 ,-1	1.8834 ,-1
.14	8.3881 ,-1	6.9423 ,-1	5.7075 ,-1	4.6739 ,-1	3.8176 ,-1	3.1124 ,-1	2.5340 ,-1	2.0611 ,-1	1.6750 ,-1	1.3604 ,-1
.16	8.1765 ,-1	6.5675 ,-1	5.2271 ,-1	4.1370 ,-1	3.2620 ,-1	2.5650 ,-1	2.0128 ,-1	1.5770 ,-1	1.2339 ,-1	9.6451 ,-2
.18	7.9694 ,-1	6.2073 ,-1	4.7762 ,-1	3.6469 ,-1	2.7698 ,-1	2.0952 ,-1	1.5802 ,-1	1.1889 ,-1	8.9260 ,-2	6.6904 ,-2
.20	7.7669 ,-1	5.8614 ,-1	4.3539 ,-1	3.2008 ,-1	2.3357 ,-1	1.6948 ,-1	1.2241 ,-1	8.8091 ,-2	6.3191 ,-2	4.5205 ,-2
.22	7.5699 ,-1	5.5296 ,-1	3.9589 ,-1	2.7959 ,-1	1.9545 ,-1	1.3554 ,-1	9.3367 ,-2	6.3949 ,-2	4.3581 ,-2	2.9565 ,-2
.24	7.3752 ,-1	5.2114 ,-1	3.5901 ,-1	2.4295 ,-1	1.6216 ,-1	1.0700 ,-1	6.9915 ,-2	4.5280 ,-2	2.9086 ,-2	1.8539 ,-2
.26	7.1860 ,-1	4.9067 ,-1	3.2465 ,-1	2.0992 ,-1	1.3323 ,-1	8.3200 ,-2	5.1199 ,-2	3.1068 ,-2	1.8593 ,-2	1.0971 ,-2
.28	7.0012 ,-1	4.6151 ,-1	2.9270 ,-1	1.8024 ,-1	1.0825 ,-1	6.3530 ,-2	3.6461 ,-2	2.0449 ,-2	1.1186 ,-2	5.9468 ,-3
.30	6.8206 ,-1	4.3363 ,-1	2.6305 ,-1	1.5369 ,-1	8.6817 ,-2	4.7444 ,-2	2.5035 ,-2	1.2691 ,-2	6.1204 ,-3	2.7549 ,-3
.32	6.6444 ,-1	4.0701 ,-1	2.3560 ,-1	1.3004 ,-1	6.8572 ,-2	3.4447 ,-2	1.6339 ,-2	7.1790 ,-3	2.7966 ,-3	8.4739 ,-4
.34	6.4724 ,-1	3.8161 ,-1	2.1025 ,-1	1.0906 ,-1	5.3167 ,-2	2.4090 ,-2	9.8700 ,-3	3.4023 ,-3	7.3840 ,-4	-1.8867 ,-4

$F_{-n}(\tau)$

τ \ n	2	3	4	5	6	7	8	9	10
0.02	1.0566 ,0	1.0844 ,0	1.1106 ,0	1.1381 ,0	1.1668 ,0	1.1963 ,0	1.2268 ,0	1.2580 ,0	1.2901 ,0
.04	1.1182 ,0	1.1900 ,0	1.2521 ,0	1.3154 ,0	1.3825 ,0	1.4540 ,0	1.5298 ,0	1.6101 ,0	1.6949 ,0
.06	1.1784 ,0	1.3128 ,0	1.4310 ,0	1.5486 ,0	1.6726 ,0	1.8061 ,0	1.9511 ,0	2.1088 ,0	2.2803 ,0
.08	1.2349 ,0	1.4460 ,0	1.6465 ,0	1.8506 ,0	2.0573 ,0	2.3039 ,0	2.5623 ,0	2.8501 ,0	3.1705 ,0
.10	1.2853 ,0	1.5815 ,0	1.8900 ,0	2.2227 ,0	2.5896 ,0	2.9996 ,0	3.4623 ,0	3.9876 ,0	4.5864 ,0
.12	1.3285 ,0	1.7113 ,0	2.1467 ,0	2.6504 ,0	3.2386 ,0	3.9293 ,0	4.7432 ,0	5.7043 ,0	6.8412 ,0
.14	1.3637 ,0	1.8275 ,0	2.3967 ,0	3.1010 ,0	3.9759 ,0	5.0644 ,0	6.4190 ,0	8.1046 ,0	1.0201 ,1
.16	1.3903 ,0	1.9233 ,0	2.6175 ,0	3.5260 ,0	4.6458 ,0	6.2817 ,0	8.3331 ,0	1.1023 ,0	1.4546 ,1
.18	1.4082 ,0	1.9928 ,0	2.7856 ,0	3.8647 ,0	5.3356 ,0	7.3413 ,0	1.0075 ,1	1.3801 ,1	1.8874 ,1
.20	1.4174 ,0	2.0312 ,0	2.8789 ,0	4.0498 ,0	5.6657 ,0	7.8909 ,0	1.0946 ,1	1.5128 ,1	2.0828 ,1
.22	1.4181 ,0	2.0351 ,0	2.8775 ,0	4.0144 ,0	5.5270 ,0	7.4993 ,0	1.0002 ,1	1.3057 ,1	1.6570 ,1
.24	1.4105 ,0	2.0023 ,0	2.7657 ,0	3.6994 ,0	4.7442 ,0	5.7174 ,0	6.1920 ,0	5.2877 ,0	1.2961 ,0
.26	1.3950 ,0	1.9318 ,0	2.5330 ,0	3.0590 ,0	3.1742 ,0	2.1608 ,0	-1.4121 ,0	-1.0271 ,1	-2.9431 ,1
.28	1.3722 ,0	1.8239 ,0	2.1743 ,0	2.0673 ,0	7.3136 ,-1	-3.4001 ,0	-1.3348 ,1	-3.4722 ,1	-7.7641 ,1
.30	1.3426 ,0	1.6799 ,0	1.6908 ,0	7.2324 ,-1	-2.5905 ,0	-1.0956 ,1	-2.9465 ,1	-6.7343 ,1	-1.4056 ,2
.32	1.3068 ,0	1.5021 ,0	1.0896 ,0	-9.5010 ,-1	-6.7052 ,0	-2.0193 ,1	-4.8731 ,1	-1.0493 ,2	-2.0930 ,2
.34	1.2655 ,0	1.2938 ,0	3.8379 ,-1	-2.9017 ,0	-1.1424 ,1	-3.0471 ,1	-6.9105 ,1	-1.4153 ,2	-2.6686 ,2

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$F_{n,1}(\tau)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	9.8091 ,-1	9.6592 ,-1	9.4773 ,-1	9.2798 ,-1	9.0757 ,-1	8.8695 ,-1	8.6636 ,-1	8.4596 ,-1	8.2582 ,-1	8.0601 ,-1
.04	9.6121 ,-1	9.3243 ,-1	8.9716 ,-1	8.5957 ,-1	8.2153 ,-1	7.8394 ,-1	7.4728 ,-1	7.1180 ,-1	6.7763 ,-1	6.4483 ,-1
.06	9.4181 ,-1	8.9954 ,-1	8.4827 ,-1	7.9468 ,-1	7.4159 ,-1	6.9033 ,-1	6.4152 ,-1	5.9545 ,-1	5.5219 ,-1	5.1172 ,-1
.08	9.2275 ,-1	8.6725 ,-1	8.0105 ,-1	7.3320 ,-1	6.6746 ,-1	6.0549 ,-1	5.4794 ,-1	4.9500 ,-1	4.4660 ,-1	4.0253 ,-1
.10	9.0382 ,-1	8.3557 ,-1	7.5548 ,-1	6.7503 ,-1	5.9887 ,-1	5.2882 ,-1	4.6545 ,-1	4.0871 ,-1	3.5825 ,-1	3.1359 ,-1
.12	8.8500 ,-1	8.0448 ,-1	7.1152 ,-1	6.2008 ,-1	5.3553 ,-1	4.5975 ,-1	3.9303 ,-1	3.3496 ,-1	2.8480 ,-1	2.4171 ,-1
.14	8.6632 ,-1	7.7399 ,-1	6.6916 ,-1	5.6824 ,-1	4.7719 ,-1	3.9773 ,-1	3.2974 ,-1	2.7229 ,-1	2.2416 ,-1	1.8410 ,-1
.16	8.4780 ,-1	7.4410 ,-1	6.2838 ,-1	5.1941 ,-1	4.2357 ,-1	3.4224 ,-1	2.7469 ,-1	2.1936 ,-1	1.7449 ,-1	1.3837 ,-1
.18	8.2943 ,-1	7.1481 ,-1	5.8916 ,-1	4.7350 ,-1	3.7443 ,-1	2.9278 ,-1	2.2705 ,-1	1.7496 ,-1	1.3415 ,-1	1.0244 ,-1
.20	8.1122 ,-1	6.8612 ,-1	5.5147 ,-1	4.3040 ,-1	3.2942 ,-1	2.4887 ,-1	1.80605 ,-1	1.3798 ,-1	1.0168 ,-1	7.4527 ,-2
.22	7.9315 ,-1	6.5804 ,-1	5.1528 ,-1	3.9003 ,-1	2.8860 ,-1	2.1007 ,-1	1.5099 ,-1	1.0744 ,-1	7.5820 ,-2	5.3128 ,-2
.24	7.7524 ,-1	6.3055 ,-1	4.8059 ,-1	3.5227 ,-1	2.5143 ,-1	1.7594 ,-1	1.2121 ,-1	8.2442 ,-2	5.5466 ,-2	3.6962 ,-2
.26	7.5747 ,-1	6.0367 ,-1	4.4735 ,-1	3.1705 ,-1	2.1778 ,-1	1.4607 ,-1	9.6094 ,-1	6.2182 ,-2	3.9656 ,-2	2.4956 ,-2
.28	7.3987 ,-1	5.7740 ,-1	4.1556 ,-1	2.8425 ,-1	1.8745 ,-1	1.2009 ,-1	7.5091 ,-2	4.5941 ,-2	2.7563 ,-2	1.6216 ,-2
.30	7.2241 ,-1	5.5172 ,-1	3.8518 ,-1	2.5380 ,-1	1.6020 ,-1	9.7628 ,-2	5.7687 ,-2	3.3117 ,-2	1.8476 ,-2	1.0005 ,-2
.32	7.0512 ,-1	5.2665 ,-1	3.5620 ,-1	2.2559 ,-1	1.3584 ,-1	7.8345 ,-2	4.3418 ,-2	2.3120 ,-2	1.1793 ,-2	5.7209 ,-3
.34	6.8799 ,-1	5.0219 ,-1	3.2858 ,-1	1.9953 ,-1	1.1416 ,-1	6.1919 ,-2	3.1857 ,-2	1.5473 ,-2	7.0062 ,-3	2.8765 ,-3

$F_{-n,1}(\tau)$

τ \ n	2	3	4	5	6	7	8	9	10
0.02	1.2228 ,0	1.2860 ,0	1.2600 ,0	1.2557 ,0	1.2684 ,0	1.2895 ,0	1.3154 ,0	1.3441 ,0	1.3748 ,0
.04	1.2521 ,0	1.5553 ,0	1.6167 ,0	1.6402 ,0	1.6737 ,0	1.7234 ,0	1.7871 ,0	1.8621 ,0	1.9461 ,0
.06	1.1915 ,0	1.7454 ,0	2.0045 ,0	2.1561 ,0	2.2775 ,0	2.4008 ,0	2.5390 ,0	2.6970 ,0	2.8762 ,0
.08	1.0875 ,0	1.8324 ,0	2.3440 ,0	2.7386 ,0	3.07878 ,0	3.4123 ,0	3.7356 ,0	4.0916 ,0	4.4827 ,0
.10	9.5180 ,-1	1.8088 ,0	2.5622 ,0	3.2708 ,0	3.9725 ,0	4.6949 ,0	5.4608 ,0	6.2912 ,0	7.2064 ,0
.12	7.9556 ,-1	1.6771 ,0	2.6013 ,0	3.6124 ,0	4.7469 ,0	6.0399 ,0	7.5282 ,0	9.2527 ,0	1.1260 ,1
.14	6.2676 ,-1	1.4458 ,0	2.4227 ,0	3.6217 ,0	5.1131 ,0	6.9783 ,0	9.3151 ,0	1.2243 ,1	1.5909 ,1
.16	4.5133 ,-1	1.1276 ,0	2.0083 ,0	3.1811 ,0	4.4880 ,0	6.8757 ,0	9.7226 ,0	1.3540 ,1	1.8646 ,1
.18	2.7410 ,-1	7.3753 ,-1	1.3594 ,0	2.2136 ,0	3.3979 ,0	5.0425 ,0	7.3225 ,0	1.0477 ,1	1.4838 ,1
.20	9.8738 ,-2	2.9209 ,-1	4.9514 ,-1	6.9372 ,-1	8.5198 ,-1	8.0533 ,-1	6.8678 ,-1	-0.16318 ,-2	-1.6240 ,0
.22	-7.1765 ,-2	-1.9159 ,-1	-5.5019 ,-1	-1.3445 ,0	-2.9251 ,0	-5.8686 ,0	-1.1154 ,1	-2.0387 ,1	-3.6196 ,1
.24	-2.3499 ,-1	-6.9648 ,-1	-1.7304 ,0	-3.7538 ,0	-7.8062 ,0	-1.5130 ,1	-2.8227 ,1	-5.1163 ,1	-9.0659 ,1
.26	-3.8915 ,-1	-1.2061 ,0	-2.9907 ,0	-6.5887 ,0	-1.3487 ,1	-2.6255 ,1	-4.9252 ,1	-8.9759 ,1	-1.5976 ,2
.28	-5.3258 ,-1	-1.7049 ,0	-4.2709 ,0	-9.4747 ,0	-1.9501 ,1	-3.8183 ,1	-7.1481 ,1	-1.2981 ,2	-2.3608 ,2
.30	-6.6410 ,-1	-2.1786 ,0	-5.5087 ,0	-1.2273 ,1	-2.5253 ,1	-4.9039 ,1	-9.0852 ,1	-1.6139 ,2	-2.7520 ,2
.32	-7.8290 ,-1	-2.6144 ,0	-6.6434 ,0	-1.4767 ,1	-3.0078 ,1	-5.7165 ,1	-1.0277 ,2	-1.7125 ,2	-2.6946 ,2
.34	-8.8830 ,-1	-3.0013 ,0	-7.6179 ,0	-1.6744 ,1	-3.3302 ,1	-6.0763 ,1	-1.0136 ,2	-1.5084 ,2	-1.7797 ,2

TABLE 2 - NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$f(\tau)T^{\nu}(\tau) \longrightarrow f(\tau)T^{\nu}(\tau)\cos\left(\nu\omega - \frac{\pi}{4}\right)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.02	9.8904 ,-1	9.6437 ,-1	9.4031 ,-1	9.1685 ,-1	8.9397 ,-1	8.7167 ,-1	8.4992 ,-1	8.2871 ,-1	8.0804 ,-1	7.8788 ,-1	7.6822 ,-1
.04	9.8148 ,-1	9.3195 ,-1	8.8492 ,-1	8.4026 ,-1	7.9786 ,-1	7.5759 ,-1	7.1936 ,-1	6.8306 ,-1	6.4859 ,-1	6.1586 ,-1	5.8478 ,-1
.06	9.7863 ,-1	9.0363 ,-1	8.3438 ,-1	7.7043 ,-1	7.1139 ,-1	6.5687 ,-1	6.0652 ,-1	5.6004 ,-1	5.1712 ,-1	4.7749 ,-1	4.4089 ,-1
.08	9.8280 ,-1	8.8098 ,-1	7.8970 ,-1	7.0789 ,-1	6.3455 ,-1	5.6881 ,-1	5.0988 ,-1	4.5705 ,-1	4.0970 ,-1	3.6726 ,-1	3.2921 ,-1
.10	9.9832 ,-1	8.6706 ,-1	7.5296 ,-1	6.5392 ,-1	5.6790 ,-1	4.9320 ,-1	4.2833 ,-1	3.7199 ,-1	3.2305 ,-1	2.8957 ,-1	2.4366 ,-1
.12	1.0349 ,0	8.6857 ,-1	7.2896 ,-1	6.1180 ,-1	5.1346 ,-1	4.3093 ,-1	3.6167 ,-1	3.0353 ,-1	2.5475 ,-1	2.1380 ,-1	1.7944 ,-1
.14	1.1215 ,0	9.0650 ,-1	7.3271 ,-1	5.9224 ,-1	4.7870 ,-1	3.8693 ,-1	3.1275 ,-1	2.5279 ,-1	2.0433 ,-1	1.6516 ,-1	1.3350 ,-1
.16	1.4373 ,0	1.1129 ,0	8.6164 ,-1	6.6714 ,-1	5.1654 ,-1	3.9901 ,-1	3.0965 ,-1	2.3957 ,-1	1.8563 ,-1	1.4373 ,-1	1.1118 ,-1
.18	1.7659 ,0	1.3027 ,0	9.6084 ,-1	7.0867 ,-1	5.2264 ,-1	3.8541 ,-1	2.8419 ,-1	2.0954 ,-1	1.5449 ,-1	1.1389 ,-1	8.3956 ,-2
.20	1.3039 ,0	9.3037 ,-1	6.6294 ,-1	4.7177 ,-1	3.3531 ,-1	2.3804 ,-1	1.6880 ,-1	1.1956 ,-1	8.4599 ,-2	5.9800 ,-2	4.2229 ,-2
.22	1.0986 ,0	7.6439 ,-1	5.2920 ,-1	3.6472 ,-1	2.5031 ,-1	1.7111 ,-1	1.1654 ,-1	7.9096 ,-2	5.3503 ,-2	3.6076 ,-2	2.4249 ,-2
.24	9.6641 ,-1	6.5845 ,-1	4.4362 ,-1	2.9597 ,-1	1.9563 ,-1	1.2825 ,-1	8.3432 ,-2	5.3874 ,-2	3.4539 ,-2	2.1988 ,-2	1.3901 ,-2
.26	8.6865 ,-1	5.8072 ,-1	3.8051 ,-1	2.4504 ,-1	1.5536 ,-1	9.7090 ,-2	5.9850 ,-2	3.6402 ,-2	2.1844 ,-2	1.2926 ,-2	7.5363 ,-3
.28	7.9084 ,-1	5.1916 ,-1	3.3028 ,-1	2.0458 ,-1	1.2371 ,01	7.3137 ,-2	4.2279 ,-2	2.3830 ,-2	1.3154 ,-2	7.0408 ,-3	3.6405 ,-3
.30	7.2598 ,-1	4.6795 ,-1	2.8838 ,-1	1.7108 ,-1	9.8029 ,-2	5.4285 ,-2	2.9000 ,-2	1.4875 ,-2	7.2570 ,-3	3.3062 ,-3	1.3515 ,-3
.32	6.7015 ,-1	4.2392 ,-1	2.5236 ,-1	1.4269 ,-1	7.6847 ,-1	3.9338 ,-2	1.8988 ,-2	8.4863 ,-3	3.3677 ,-3	1.0490 ,-3	1.0318 ,-3
.34	6.2099 ,-1	3.8516 ,-1	2.2087 ,-1	1.1831 ,-1	5.9276 ,-1	2.7515 ,-2	1.1559 ,-2	4.0737 ,-3	9.2323 ,-4	2.0610 ,-4	-4.7533 ,-4

$f(\tau)T^{-\nu}(\tau) \longrightarrow \frac{1}{2}f(\tau)T^{-\nu}(\tau)\cos\left(\nu\omega + \frac{\pi}{4}\right)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.02	1.0144 ,0	1.0403 ,0	1.0669 ,0	1.0942 ,0	1.1222 ,0	1.1509 ,0	1.1804 ,0	1.2106 ,0	1.2416 ,0	1.2733 ,0	1.3059 ,0
.04	1.0336 ,0	1.0886 ,0	1.1464 ,0	1.2074 ,0	1.2715 ,0	1.3391 ,0	1.4103 ,0	1.4852 ,0	1.5642 ,0	1.6473 ,0	1.7348 ,0
.06	1.0599 ,0	1.1478 ,0	1.2431 ,0	1.3463 ,0	1.4580 ,0	1.5790 ,0	1.7101 ,0	1.8521 ,0	2.0058 ,0	2.1723 ,0	2.3526 ,0
.08	1.0964 ,0	1.2231 ,0	1.3644 ,0	1.5222 ,0	1.6981 ,0	1.8944 ,0	2.1133 ,0	2.3576 ,0	2.6300 ,0	2.9340 ,0	3.2731 ,0
.10	1.1495 ,0	1.3236 ,0	1.5241 ,0	1.7549 ,0	2.0207 ,0	2.3268 ,0	2.6792 ,0	3.0850 ,0	3.5522 ,0	4.0903 ,0	4.7098 ,0
.12	1.2332 ,0	1.4693 ,0	1.7507 ,0	2.0860 ,0	2.4855 ,0	2.9615 ,0	3.5286 ,0	4.2044 ,0	5.0096 ,0	5.9691 ,0	7.1122 ,0
.14	1.3875 ,0	1.7166 ,0	2.1237 ,0	2.6274 ,0	3.2506 ,0	4.0216 ,0	4.9755 ,0	6.1556 ,0	7.6155 ,0	9.4218 ,0	1.1657 ,1
.16	1.8564 ,0	2.3976 ,0	3.0967 ,0	3.9996 ,0	5.1657 ,0	6.6717 ,0	8.6169 ,0	1.1129 ,1	1.4374 ,1	1.8565 ,1	2.3978 ,1
.18	1.1969 ,0	1.6223 ,0	2.1986 ,0	2.9796 ,0	4.0375 ,0	5.4707 ,0	7.4119 ,0	1.0041 ,1	1.3601 ,1	1.8423 ,1	2.4951 ,1
.20	9.1239 ,-1	1.2749 ,0	1.7787 ,0	2.4773 ,0	3.4440 ,0	4.7786 ,0	6.6166 ,0	9.1409 ,0	1.2598 ,1	1.7315 ,1	2.3731 ,1
.22	7.8702 ,-1	1.1156 ,0	1.5742 ,0	2.2040 ,0	3.0578 ,0	4.1969 ,0	5.6861 ,0	7.5803 ,0	9.9041 ,0	1.2573 ,1	1.5354 ,1
.24	7.0009 ,-1	9.9892 ,-1	1.3989 ,0	1.9132 ,0	2.5353 ,0	3.2122 ,0	3.7935 ,0	3.9393 ,0	2.9542 ,0	-4.9997 ,-1	-2.8145 ,0
.26	6.3416 ,-1	8.9806 ,0	1.2205 ,0	1.5613 ,0	1.8045 ,0	1.6807 ,0	6.2377 ,-1	-2.4861 ,0	-9.7745 ,0	-2.5154 ,1	-5.5657 ,1
.28	5.7915 ,-1	8.0599 ,0	1.0294 ,0	1.1360 ,0	8.4266 ,-1	-4.5290 ,-1	-3.9635 ,0	-1.2026 ,1	-2.8994 ,1	-6.2752 ,1	-1.2723 ,2
.30	5.3124 ,-1	7.1373 ,0	8.2380 ,-1	6.3888 ,-1	-3.4283 ,-1	-3.1687 ,0	-9.9092 ,0	-2.4475 ,1	-5.4054 ,1	-1.1123 ,2	-2.1752 ,2
.32	4.8840 ,-1	6.2691 ,0	6.0480 ,-1	7.7970 ,-2	-1.7241 ,0	-6.3792 ,0	-1.6945 ,1	-3.9055 ,1	-8.2602 ,1	-1.6406 ,2	-3.0919 ,2
.34	4.4941 ,-1	5.3883 ,0	3.7491 ,-1	-5.3405 ,-1	-3.2552 ,0	-9.9315 ,0	-2.3478 ,1	-5.4370 ,1	-1.1046 ,2	-2.0945 ,2	-3.7393 ,2

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$q(\tau)T^{\nu} \longrightarrow q(\tau)T^{\nu} \cos\left(\nu\omega - \mu^{\circ} - \frac{\pi}{4}\right)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.02	1.0360 ,0	1.0102 ,0	9.8498 ,-1	9.6040 ,-1	9.3644 ,-1	9.1307 ,-1	8.9029 ,-1	8.6808 ,-1	8.4642 ,-1	8.2530 ,-1	8.0471 ,-1
.04	1.0812 ,0	1.0266 ,0	9.7481 ,-1	9.2562 ,-1	8.7891 ,-1	8.3455 ,-1	7.9244 ,-1	7.5245 ,-1	7.1447 ,-1	6.7842 ,-1	6.4418 ,-1
.06	1.1394 ,0	1.0520 ,0	9.7141 ,-1	8.9696 ,-1	8.2822 ,-1	7.6476 ,-1	7.0613 ,-1	6.5202 ,-1	6.0205 ,-1	5.5591 ,-1	5.1330 ,-1
.08	1.2171 ,0	1.0910 ,0	9.7800 ,-1	8.7667 ,-1	7.8585 ,-1	7.0443 ,-1	6.3145 ,-1	5.6603 ,-1	5.0739 ,-1	4.5482 ,-1	4.0770 ,-1
.10	1.3270 ,0	1.1525 ,0	1.0009 ,0	8.6922 ,-1	7.5489 ,-1	6.5559 ,-1	5.6936 ,-1	4.9447 ,-1	4.2943 ,-1	3.7294 ,-1	3.2389 ,-1
.12	1.4967 ,0	1.2562 ,0	1.0543 ,0	8.8480 ,-1	7.4258 ,-1	6.2322 ,-1	5.2305 ,-1	4.3898 ,-1	3.6842 ,-1	3.0920 ,-1	2.5950 ,-1
.14	1.8077 ,0	1.4611 ,0	1.1810 ,0	9.5459 ,-1	7.7158 ,-1	6.2366 ,-1	5.0410 ,-1	4.0746 ,-1	3.2934 ,-1	2.6621 ,-1	2.1517 ,-1
.16	2.7517 ,0	2.1305 ,0	1.6496 ,0	1.2772 ,0	9.8888 ,-1	7.6565 ,-1	5.9281 ,-1	4.5899 ,-1	3.5537 ,-1	2.7515 ,-1	2.1304 ,-1
.18	2.8287 ,0	2.0973 ,0	1.5547 ,0	1.1524 ,0	8.5398 ,-1	6.3277 ,-1	4.6878 ,-1	3.4724 ,-1	2.5717 ,-1	1.9044 ,-1	1.4100 ,-1
.20	1.3958 ,0	1.0429 ,0	7.7499 ,-1	5.7312 ,-1	4.2203 ,-1	3.0959 ,-1	2.2634 ,-1	1.6497 ,-1	1.1990 ,-1	8.6915 ,-2	6.2858 ,-2
.22	8.2101 ,-1	6.5668 ,-1	5.0863 ,-1	3.8484 ,-1	2.8598 ,-1	2.0951 ,-1	1.5171 ,-1	1.0878 ,-1	7.7352 ,-2	5.4606 ,-2	3.8302 ,-2
.24	5.0887 ,-1	4.6596 ,-1	3.8448 ,-1	2.9866 ,-1	2.2293 ,-1	1.6170 ,-1	1.1474 ,-1	8.0010 ,-2	5.4989 ,-2	3.7329 ,-2	2.5069 ,-2
.26	3.1996 ,-1	3.6030 ,-1	3.1755 ,-1	2.5074 ,-1	1.8569 ,-1	1.3173 ,-1	9.0534 ,-2	6.0697 ,-2	3.9867 ,-2	2.5727 ,-2	1.6343 ,-2
.28	1.9919 ,-1	2.9781 ,-1	2.7736 ,-1	2.1953 ,-1	1.5937 ,-1	1.0942 ,-1	7.2138 ,-2	4.6055 ,-2	2.8615 ,-2	1.7353 ,-2	1.0287 ,-2
.30	1.1952 ,-1	2.5898 ,-1	2.5060 ,-1	1.9628 ,-1	1.3834 ,-1	9.1165 ,-2	5.7179 ,-2	3.4444 ,-2	2.0020 ,-2	1.1246 ,-2	6.1005 ,-3
.32	6.6117 ,-2	2.3381 ,-1	2.3092 ,-1	1.7714 ,-1	1.2030 ,-1	7.5557 ,-2	4.4721 ,-2	2.5154 ,-2	1.3477 ,-2	6.8593 ,-3	3.2876 ,-3
.34	3.0144 ,-2	2.1676 ,-1	2.1512 ,-1	1.6033 ,-1	1.0427 ,-1	6.1989 ,-2	3.4310 ,-2	1.7773 ,-2	8.5807 ,-3	3.7992 ,-3	1.4796 ,-3

$q(\tau)T^{-\nu} \longrightarrow \frac{1}{2}qT^{-\nu} \cos\left(\nu\omega + \mu^{\circ} + \frac{\pi}{4}\right)$

τ \ ν	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
0.02	1.0625 ,0	1.0897 ,0	1.1176 ,0	1.1462 ,0	1.1755 ,0	1.7056 ,0	1.2365 ,0	1.2681 ,0	1.3006 ,0	1.3338 ,0	1.3680 ,0
.04	1.1386 ,0	1.1992 ,0	1.2629 ,0	1.3300 ,0	1.4007 ,0	1.4751 ,0	1.5535 ,0	1.6361 ,0	1.7231 ,0	1.8146 ,0	1.9111 ,0
.06	1.2339 ,0	1.3363 ,0	1.4473 ,0	1.5674 ,0	1.6975 ,0	1.8384 ,0	1.9910 ,0	2.1562 ,0	2.3352 ,0	2.5290 ,0	2.7389 ,0
.08	1.3578 ,0	1.5147 ,0	1.6898 ,0	1.8851 ,0	2.1030 ,0	2.3460 ,0	2.6172 ,0	2.9197 ,0	3.2571 ,0	3.6336 ,0	4.0535 ,0
.10	1.5280 ,0	1.7594 ,0	2.0259 ,0	2.3328 ,0	2.6861 ,0	3.0929 ,0	3.5614 ,0	4.1007 ,0	4.7218 ,0	5.4370 ,0	6.2605 ,0
.12	1.7834 ,0	2.1249 ,0	2.5319 ,0	3.0168 ,0	3.5945 ,0	4.2829 ,0	5.1032 ,0	6.0805 ,0	7.2450 ,0	8.6326 ,0	1.0286 ,1
.14	2.2364 ,0	2.7668 ,0	3.4231 ,0	4.2349 ,0	5.2394 ,0	6.4821 ,0	8.0195 ,0	9.9216 ,0	1.2275 ,1	1.5186 ,1	1.8788 ,1
.16	3.5540 ,0	4.5901 ,0	5.9284 ,0	7.6569 ,0	9.8893 ,0	1.2773 ,1	1.6497 ,1	2.1306 ,1	2.7518 ,1	3.5541 ,1	4.5904 ,1
.18	1.9072 ,0	2.5714 ,0	3.4663 ,0	4.6718 ,0	6.2953 ,0	8.4813 ,0	1.1424 ,1	1.5385 ,1	2.0715 ,1	2.7885 ,1	3.7529 ,1
.20	9.2815 ,-1	1.2252 ,0	1.6031 ,0	2.0754 ,0	2.6516 ,0	3.3316 ,0	4.0952 ,0	4.8841 ,0	5.574 ,0	5.9239 ,0	5.5058 ,0
.22	4.8961 ,-1	5.4098 ,-1	5.1581 ,-1	3.2310 ,-1	-2.0003 ,-1	-1.3374 ,0	-3.5733 ,0	-7.7206 ,0	-1.5127 ,1	-2.8001 ,1	-4.9932 ,1
.24	2.2068 ,-1	6.1493 ,-2	-3.5453 ,-1	-1.2656 ,0	-3.0937 ,0	-6.5720 ,0	-1.2956 ,1	-2.4368 ,1	-4.4354 ,1	-7.8783 ,1	-1.3778 ,2
.26	3.6819 ,-2	-3.1011 ,-1	-1.1091 ,0	-2.7788 ,0	-6.0697 ,0	-1.2301 ,1	-2.3749 ,1	-4.4290 ,1	-8.0421 ,1	-1.4289 ,2	-2.4924 ,2
.28	-9.5925 ,-2	-6.1109 ,-1	-1.7787 ,0	-4.2161 ,0	-9.0383 ,0	-1.8209 ,1	-3.5108 ,1	-6.5426 ,1	-1.1853 ,2	-2.0944 ,2	-3.6166 ,2
.30	-1.9469 ,-1	-8.5953 ,-1	-2.3725 ,0	-5.5496 ,0	-1.1858 ,1	-2.3850 ,1	-4.5809 ,1	-8.4651 ,1	-1.5105 ,2	-2.6049 ,2	-4.3352 ,2
.32	-2.6939 ,-1	-1.0658 ,0	-2.8929 ,0	-6.7446 ,0	-1.4375 ,1	-2.8724 ,1	-5.4422 ,1	-9.8189 ,1	-1.6856 ,2	-2.7361 ,2	-4.1374 ,2
.34	-3.2630 ,-1	-1.2367 ,0	-3.3392 ,0	-7.7663 ,0	-1.6435 ,1	-3.2320 ,1	-5.9472 ,1	-1.0213 ,2	-1.6144 ,2	-2.2663 ,2	-2.5464 ,2

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Continued

$f(\tau)T^{n} \longrightarrow f(\tau)T^{n} \cos\left(n\omega - \frac{\pi}{4}\right)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	9.7663 ,-1	9.5226 ,-1	9.2850 ,-1	9.0534 ,-1	8.8275 ,-1	8.6073 ,-1	8.3925 ,-1	8.1831 ,-1	7.9789 ,-1	7.7799 ,-1
.04	9.5639 ,-1	9.0813 ,-1	8.6230 ,-1	8.1878 ,-1	7.7746 ,-1	7.3823 ,-1	7.0097 ,-1	6.6560 ,-1	6.3201 ,-1	6.0011 ,-1
.06	9.4039 ,-1	8.6932 ,-1	8.0177 ,-1	7.4032 ,-1	6.8358 ,-1	6.3119 ,-1	5.8282 ,-1	5.3815 ,-1	4.9691 ,-1	4.5882 ,-1
.08	9.3050 ,-1	8.3409 ,-1	7.4768 ,-1	6.7022 ,-1	6.0078 ,-1	5.3854 ,-1	4.8274 ,-1	4.3273 ,-1	3.8790 ,-1	3.4771 ,-1
.10	9.3034 ,-1	8.0797 ,-1	7.0169 ,-1	6.0939 ,-1	5.2924 ,-1	4.5962 ,-1	3.9917 ,-1	3.4666 ,-1	3.0106 ,-1	2.6146 ,-1
.12	9.4810 ,-1	7.9571 ,-1	6.6781 ,-1	5.6047 ,-1	4.7039 ,-1	3.9478 ,-1	3.3133 ,-1	2.7807 ,-1	2.3338 ,-1	1.9587 ,-1
.14	1.0083 ,0	8.1498 ,-1	6.5874 ,-1	5.3245 ,-1	4.3038 ,-1	3.4787 ,-1	2.8118 ,-1	2.2727 ,-1	1.8370 ,-1	1.4849 ,-1
.16	1.2647 ,0	9.7923 ,-1	7.5818 ,-1	5.8703 ,-1	4.5451 ,-1	3.5191 ,-1	2.7247 ,-1	2.1096 ,-1	1.6334 ,-1	1.2647 ,-1
.18	1.5167 ,0	1.1188 ,0	8.2518 ,-1	6.0859 ,-1	4.4881 ,-1	3.3095 ,-1	2.4403 ,-1	1.7992 ,-1	1.3265 ,-1	9.7784 ,-2
.20	1.1016 ,0	7.8549 ,-1	5.5933 ,-1	4.0799 ,-1	2.8256 ,-1	2.0048 ,-1	1.4208 ,-1	1.0060 ,-1	7.1143 ,-2	5.0258 ,-2
.22	9.1698 ,-1	6.3639 ,-1	4.3957 ,-1	3.0230 ,-1	2.0705 ,-1	1.4128 ,-1	9.6050 ,-2	6.5080 ,-2	4.3951 ,-2	2.7730 ,-2
.24	7.9889 ,-1	5.4117 ,-1	3.6374 ,-1	2.4086 ,-1	1.5856 ,-1	1.0354 ,-1	6.7104 ,-2	4.3173 ,-2	2.7582 ,-2	1.7498 ,-2
.26	7.1218 ,-1	4.7116 ,-1	3.0597 ,-1	1.9547 ,-1	1.2303 ,-1	7.6358 ,-2	4.6755 ,-2	2.8233 ,-2	1.6834 ,-2	9.8894 ,-3
.28	6.4353 ,-1	4.1557 ,-1	2.6076 ,-1	1.5955 ,-1	9.5388 ,-2	5.5765 ,-2	3.1870 ,-2	1.7782 ,-2	9.6610 ,-3	5.0874 ,-3
.30	5.8651 ,-1	3.6922 ,-1	2.2312 ,-1	1.3005 ,-1	7.3264 ,-2	3.9864 ,-2	2.0885 ,-2	1.0464 ,-2	4.9489 ,-3	2.1515 ,-3
.32	5.3755 ,-1	3.2932 ,-1	1.9093 ,-1	1.0536 ,-1	5.5351 ,-2	2.7554 ,-2	1.2839 ,-2	5.4499 ,-2	1.9678 ,-3	4.6440 ,-4
.34	4.9449 ,-1	2.9423 ,-1	1.6297 ,-1	8.4475 ,-2	4.0817 ,-2	1.8091 ,-2	7.0579 ,-3	2.1371 ,-2	1.9764 ,-4	-4.0399 ,-4

$f(\tau)T^{-n} \longrightarrow \frac{1}{2}f(\tau)T^{-n} \cos\left(n\omega + \frac{\pi}{4}\right)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	1.0273 ,0	1.0535 ,0	1.0805 ,0	1.1081 ,0	1.1365 ,0	1.1656 ,0	1.1954 ,0	1.2260 ,0	1.2574 ,0	1.2895 ,0
.04	1.0608 ,0	1.1171 ,0	1.1765 ,0	1.2390 ,0	1.3049 ,0	1.3742 ,0	1.4473 ,0	1.5242 ,0	1.6052 ,0	1.6905 ,0
.06	1.1030 ,0	1.1945 ,0	1.2937 ,0	1.4010 ,0	1.5173 ,0	1.6433 ,0	1.7797 ,0	1.9274 ,0	2.0874 ,0	2.2606 ,0
.08	1.1580 ,0	1.2919 ,0	1.4412 ,0	1.6077 ,0	1.7935 ,0	2.0008 ,0	2.2321 ,0	2.4901 ,0	2.7779 ,0	3.0989 ,0
.10	1.2335 ,0	1.4203 ,0	1.6354 ,0	1.8831 ,0	2.1684 ,0	2.4968 ,0	2.8749 ,0	3.3104 ,0	3.8118 ,0	4.3891 ,0
.12	1.3460 ,0	1.6038 ,0	1.9110 ,0	2.2770 ,0	2.7130 ,0	3.2326 ,0	3.8517 ,0	4.5894 ,0	5.4683 ,0	6.5156 ,0
.14	1.5433 ,0	1.9093 ,0	2.3622 ,0	2.9225 ,0	3.6156 ,0	4.4732 ,0	5.5341 ,0	6.8467 ,0	8.4706 ,0	1.0480 ,1
.16	2.1097 ,0	2.7248 ,0	3.5193 ,0	4.54537 ,0	5.8706 ,0	7.5822 ,0	9.7929 ,0	1.2648 ,1	1.6336 ,1	2.1099 ,1
.18	1.3934 ,0	1.8886 ,0	2.5595 ,0	3.4685 ,0	4.6998 ,0	6.3678 ,0	8.6269 ,0	1.1687 ,1	1.5830 ,1	2.1440 ,1
.20	1.0646 ,0	1.5062 ,0	2.0996 ,0	2.9216 ,0	4.0578 ,0	5.6245 ,0	7.7792 ,0	1.0725 ,1	1.4770 ,1	2.0278 ,1
.22	9.3669 ,-1	1.3264 ,0	1.8646 ,0	2.5992 ,0	3.5876 ,0	4.8939 ,0	6.5800 ,0	8.6878 ,0	1.1201 ,1	1.3976 ,1
.24	8.3802 ,-1	1.1852 ,0	1.6415 ,0	2.2126 ,0	2.8732 ,0	3.5301 ,0	3.9533 ,0	3.6520 ,0	1.6561 ,0	-3.8627 ,0
.26	7.5795 ,-1	1.0533 ,0	1.3934 ,0	1.7070 ,0	1.8146 ,0	1.3218 ,0	-5.7242 ,-1	-5.4175 ,0	-1.6108 ,1	-3.7901 ,1
.28	6.8787 ,-1	9.2160 ,-1	1.1166 ,0	1.0669 ,0	3.7702 ,-1	-1.8198 ,0	-7.2150 ,0	-1.9010 ,1	-4.3084 ,1	-8.9945 ,1
.30	6.2405 ,-1	7.8731 ,-1	7.8725 ,-1	2.9601 ,-1	-1.4243 ,0	-5.8608 ,0	-1.5868 ,1	-3.6768 ,1	-7.8105 ,1	-1.5642 ,2
.32	5.6463 ,-1	6.4990 ,-1	4.3589 ,-1	-5.9016 ,-1	-3.5490 ,0	-1.0648 ,1	-2.6062 ,1	-5.7304 ,1	-1.1736 ,2	-2.2662 ,2
.34	5.0860 ,-1	5.0997 ,-1	6.3203 ,-2	-1.5671 ,0	-5.9035 ,0	-1.5907 ,1	-3.6952 ,1	-7.8132 ,1	-1.5354 ,2	-2.8234 ,2

TABLE 2.- NUMERICAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND THEIR ASYMPTOTIC REPRESENTATION - Concluded

$q(\tau)T^{n}(\tau) \longrightarrow q(\tau)T^{n}(\tau)\cos\left(n\omega - \mu^{\circ} - \frac{\pi}{4}\right)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	1.0230 ,0	9.9750 ,-1	9.7261 ,-1	9.4834 ,-1	9.2468 ,-1	9.0161 ,-1	8.7912 ,-1	8.5718 ,-1	8.3580 ,-1	8.1494 ,-1
.04	1.0535 ,0	1.0003 ,0	9.4986 ,-1	9.0193 ,-1	8.5641 ,-1	8.1319 ,-1	7.7215 ,-1	7.3319 ,-1	6.9619 ,-1	6.6105 ,-1
.06	1.0948 ,0	1.0109 ,0	9.3344 ,-1	8.6190 ,-1	7.9585 ,-1	7.3485 ,-1	6.7853 ,-1	6.2653 ,-1	5.7851 ,-1	5.3418 ,-1
.08	1.1523 ,0	1.0329 ,0	9.2592 ,-1	8.2999 ,-1	7.4400 ,-1	6.6692 ,-1	5.9783 ,-1	5.3589 ,-1	4.8037 ,-1	4.3060 ,-1
.10	1.2367 ,0	1.0740 ,0	9.3275 ,-1	8.1006 ,-1	7.0351 ,-1	6.1097 ,-1	5.3061 ,-1	4.6081 ,-1	4.0020 ,-1	3.4756 ,-1
.12	1.3712 ,0	1.1508 ,0	9.6583 ,-1	8.1059 ,-1	6.8030 ,-1	5.7095 ,-1	4.7918 ,-1	4.0216 ,-1	3.3752 ,-1	2.8327 ,-1
.14	1.6251 ,0	1.3136 ,0	1.0618 ,0	8.5821 ,-1	6.9368 ,-1	5.6069 ,-1	4.5320 ,-1	3.6632 ,-1	2.9609 ,-1	2.3933 ,-1
.16	2.4213 ,0	1.8747 ,0	1.4515 ,0	1.1238 ,0	8.7014 ,-1	6.7371 ,-1	5.2163 ,-1	4.0387 ,-1	3.1270 ,-1	2.4211 ,-1
.18	2.4357 ,0	1.8058 ,0	1.3385 ,0	9.9205 ,-1	7.3511 ,-1	5.4465 ,-1	4.0347 ,-1	2.9884 ,-1	2.2131 ,-1	1.6389 ,-1
.20	1.2075 ,0	8.9962 ,-1	6.6684 ,-1	4.9205 ,-1	3.6164 ,-1	2.6482 ,-1	1.9330 ,-1	1.4072 ,-1	1.0197 ,-1	7.3952 ,-2
.22	7.3776 ,-1	5.7990 ,-1	4.4355 ,-1	3.3241 ,-1	2.4517 ,-1	1.7852 ,-1	1.2838 ,-1	9.1820 ,-2	6.5046 ,-2	4.5767 ,-2
.24	4.9558 ,-1	4.2729 ,-1	3.4090 ,-1	2.5912 ,-1	1.9046 ,-1	1.3654 ,-1	9.6007 ,-2	6.6442 ,-2	4.5373 ,-2	3.0631 ,-2
.26	3.5621 ,-1	3.4457 ,-1	2.8501 ,-1	2.1716 ,-1	1.5711 ,-1	1.0958 ,-1	7.4512 ,-2	4.9307 ,-2	3.2056 ,-2	2.0543 ,-2
.28	2.7133 ,-1	2.9588 ,-1	2.5019 ,-1	1.8862 ,-1	1.3282 ,-1	8.9240 ,-2	5.7850 ,-2	3.6418 ,-2	2.2349 ,-2	1.3398 ,-2
.30	2.1779 ,-1	2.6497 ,-1	2.2561 ,-1	1.6645 ,-1	1.1309 ,-1	7.2595 ,-2	4.4586 ,-2	2.6373 ,-2	1.5068 ,-2	8.5508 ,-3
.32	1.8327 ,-1	2.4386 ,-1	2.0631 ,-1	1.4769 ,-1	9.6129 ,-2	5.8510 ,-2	3.3745 ,-2	1.8525 ,-2	9.6802 ,-3	4.7888 ,-3
.34	1.6073 ,-1	2.2830 ,-1	1.8988 ,-1	1.3100 ,-1	8.1177 ,-2	4.6506 ,-2	2.4900 ,-2	1.2466 ,-2	5.7815 ,-3	2.4201 ,-3

$q(\tau)T^{-n}(\tau) \longrightarrow \frac{1}{2}q(\tau)T^{-n}(\tau)\cos\left(n\omega + \mu^{\circ} + \frac{\pi}{4}\right)$

τ \ n	1	2	3	4	5	6	7	8	9	10
0.02	1.0760 ,0	1.1036 ,0	1.1318 ,0	1.1608 ,0	1.1905 ,0	1.2209 ,0	1.2522 ,0	1.2842 ,0	1.3171 ,0	1.3508 ,0
.04	1.1685 ,0	1.2306 ,0	1.2960 ,0	1.3648 ,0	1.4374 ,0	1.5138 ,0	1.5942 ,0	1.6790 ,0	1.7682 ,0	1.8622 ,0
.06	1.2841 ,0	1.3907 ,0	1.5061 ,0	1.6311 ,0	1.7665 ,0	1.9131 ,0	2.0719 ,0	2.2439 ,0	2.4302 ,0	2.6319 ,0
.08	1.4341 ,0	1.5998 ,0	1.7847 ,0	1.9910 ,0	2.2211 ,0	2.4778 ,0	2.7642 ,0	3.0837 ,0	3.4401 ,0	3.8377 ,0
.10	1.6397 ,0	1.8880 ,0	2.1740 ,0	2.5023 ,0	2.8824 ,0	3.3189 ,0	3.8216 ,0	4.4004 ,0	5.0669 ,0	5.8343 ,0
.12	1.9467 ,0	2.3195 ,0	2.7637 ,0	3.2930 ,0	3.9237 ,0	4.6752 ,0	5.5705 ,0	6.6374 ,0	7.9086 ,0	9.4232 ,0
.14	2.4875 ,0	3.0775 ,0	3.8074 ,0	4.7104 ,0	5.8276 ,0	7.2098 ,0	8.9199 ,0	1.1036 ,1	1.3653 ,1	1.6891 ,1
.16	4.0390 ,0	5.2165 ,0	6.7375 ,0	8.7018 ,0	1.1239 ,1	1.4516 ,1	1.8748 ,1	2.4214 ,1	3.1274 ,1	4.0392 ,1
.18	2.2855 ,0	2.9856 ,0	4.0243 ,0	5.4232 ,0	7.2911 ,0	9.8436 ,0	1.3258 ,1	1.7852 ,1	2.4035 ,1	3.2350 ,1
.20	1.0675 ,0	1.4032 ,0	1.8267 ,0	2.3016 ,0	2.9792 ,0	3.7055 ,0	4.4924 ,0	5.2591 ,0	5.8111 ,0	5.8602 ,0
.22	5.2108 ,-1	5.4240 ,-1	4.4843 ,-1	1.16998 ,-1	-6.6758 ,-1	-2.2768 ,0	-5.3394 ,0	-1.0903 ,1	-2.0696 ,1	-4.1599 ,1
.24	1.6313 ,-1	-1.0276 ,-1	-7.2736 ,-1	-2.0286 ,0	-4.5637 ,0	-9.2942 ,0	-1.7855 ,1	-3.2992 ,1	-5.9273 ,1	-1.0422 ,2
.26	-9.7673 ,-2	-6.3264 ,-1	-1.7981 ,0	-4.1561 ,0	-8.7037 ,0	-1.7178 ,1	-3.2505 ,1	-5.9859 ,1	-1.0746 ,2	-1.8912 ,2
.28	-2.9762 ,-1	-1.0833 ,0	-2.7838 ,0	-6.2201 ,0	-1.2910 ,1	-2.5404 ,1	-4.8110 ,1	-8.8350 ,1	-1.5803 ,2	-2.75996 ,2
.30	-4.5501 ,-1	-1.4704 ,0	-3.6807 ,0	-8.1849 ,0	-1.6926 ,1	-3.3226 ,1	-6.2554 ,1	-1.1356 ,2	-1.9958 ,2	-3.3761 ,2
.32	-5.8031 ,-1	-1.8021 ,0	-4.4782 ,0	-9.9375 ,0	-2.0758 ,1	-3.9786 ,1	-7.3545 ,1	-1.2680 ,2	-2.1641 ,2	-3.3994 ,2
.34	-6.8054 ,-1	-2.0831 ,0	-5.1630 ,0	-1.1412 ,1	-2.3244 ,1	-4.4205 ,1	-7.8654 ,1	-1.3009 ,2	-1.9487 ,2	-2.5018 ,2

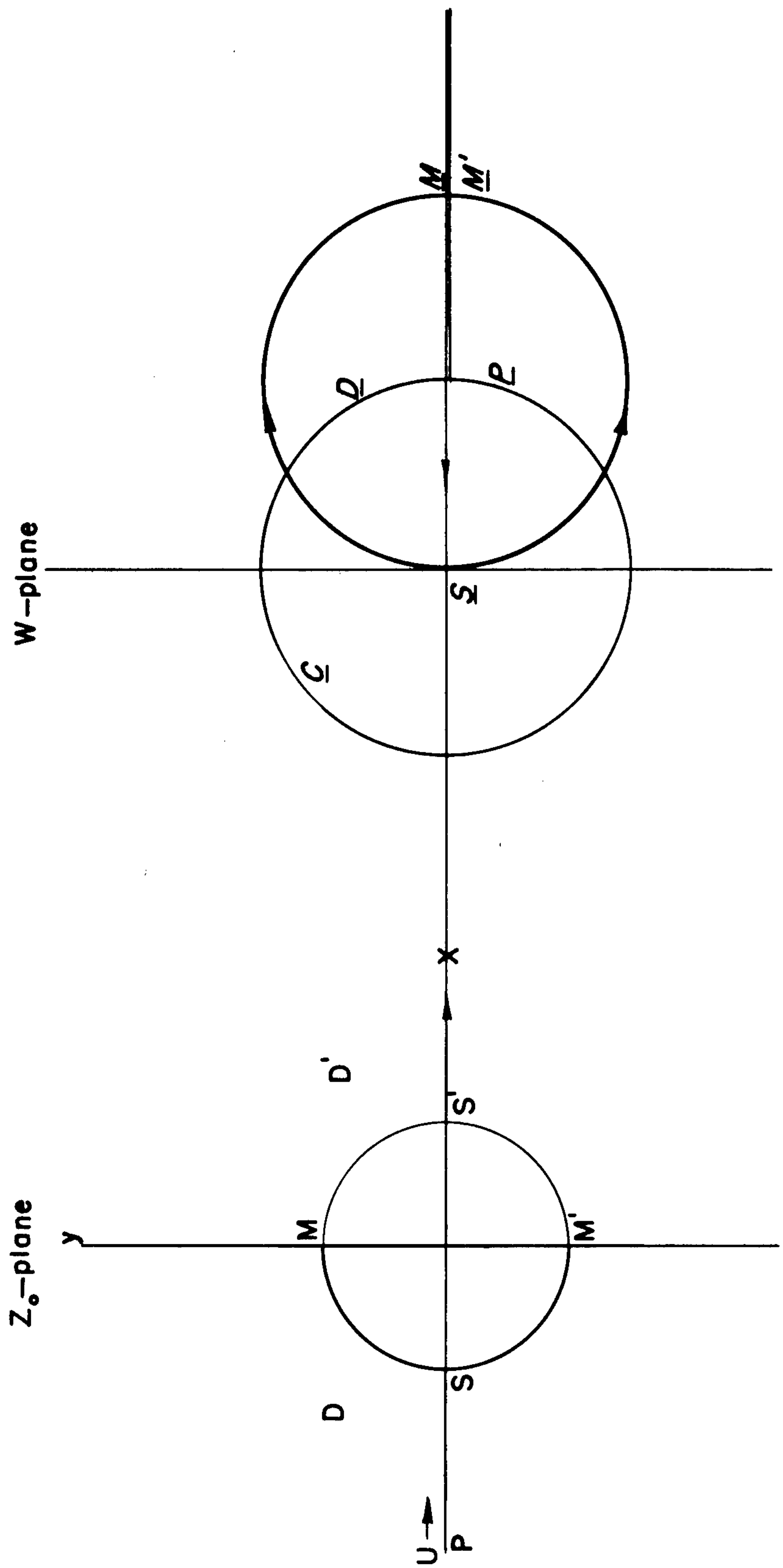

Figure 1.- Mapping of half plane D.

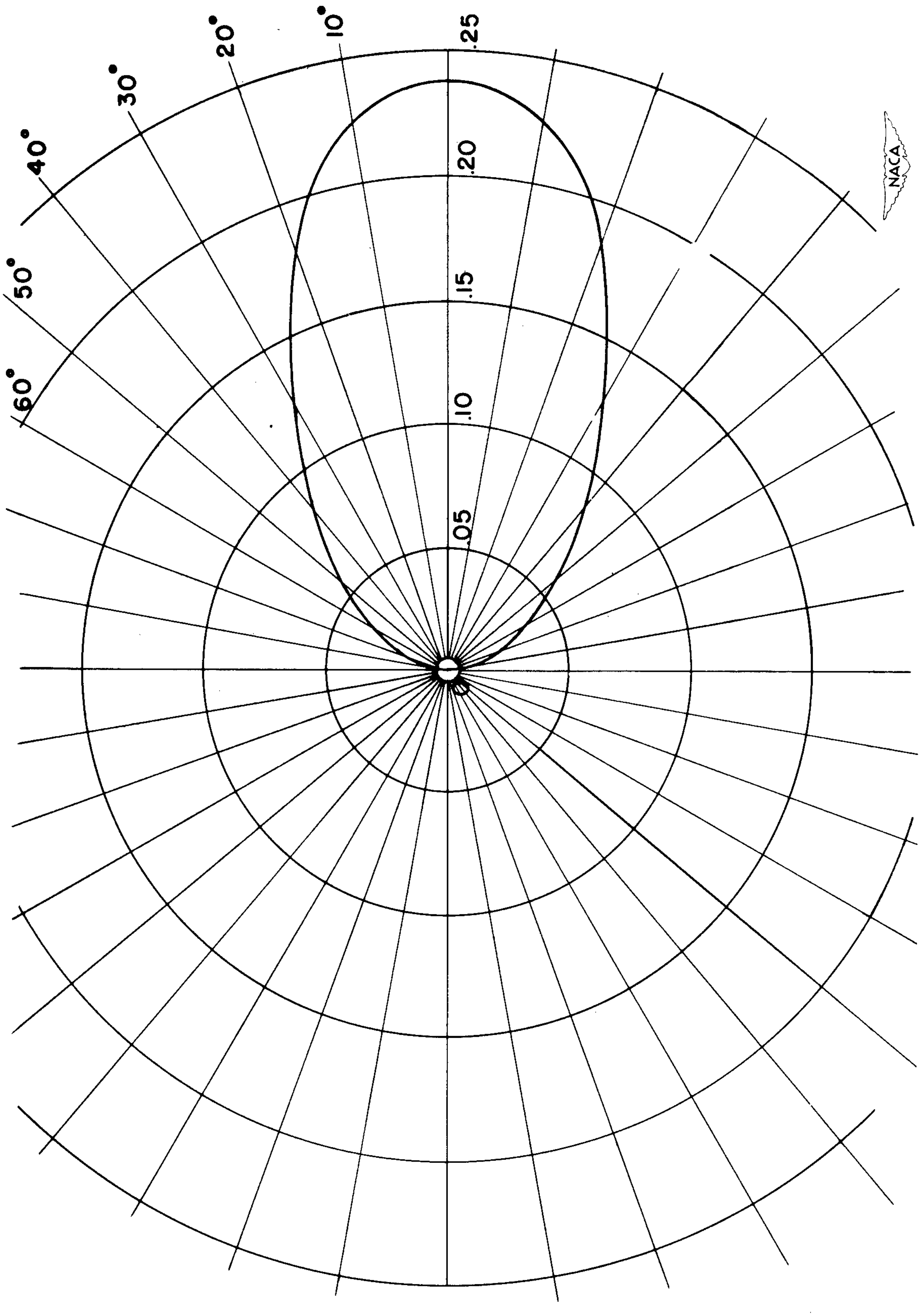

Figure 2.- Zero streamline of a compressible flow in hodograph plane. $\epsilon = \frac{1}{2}$; $M_1 = 0.60$; $r_o = 0$.

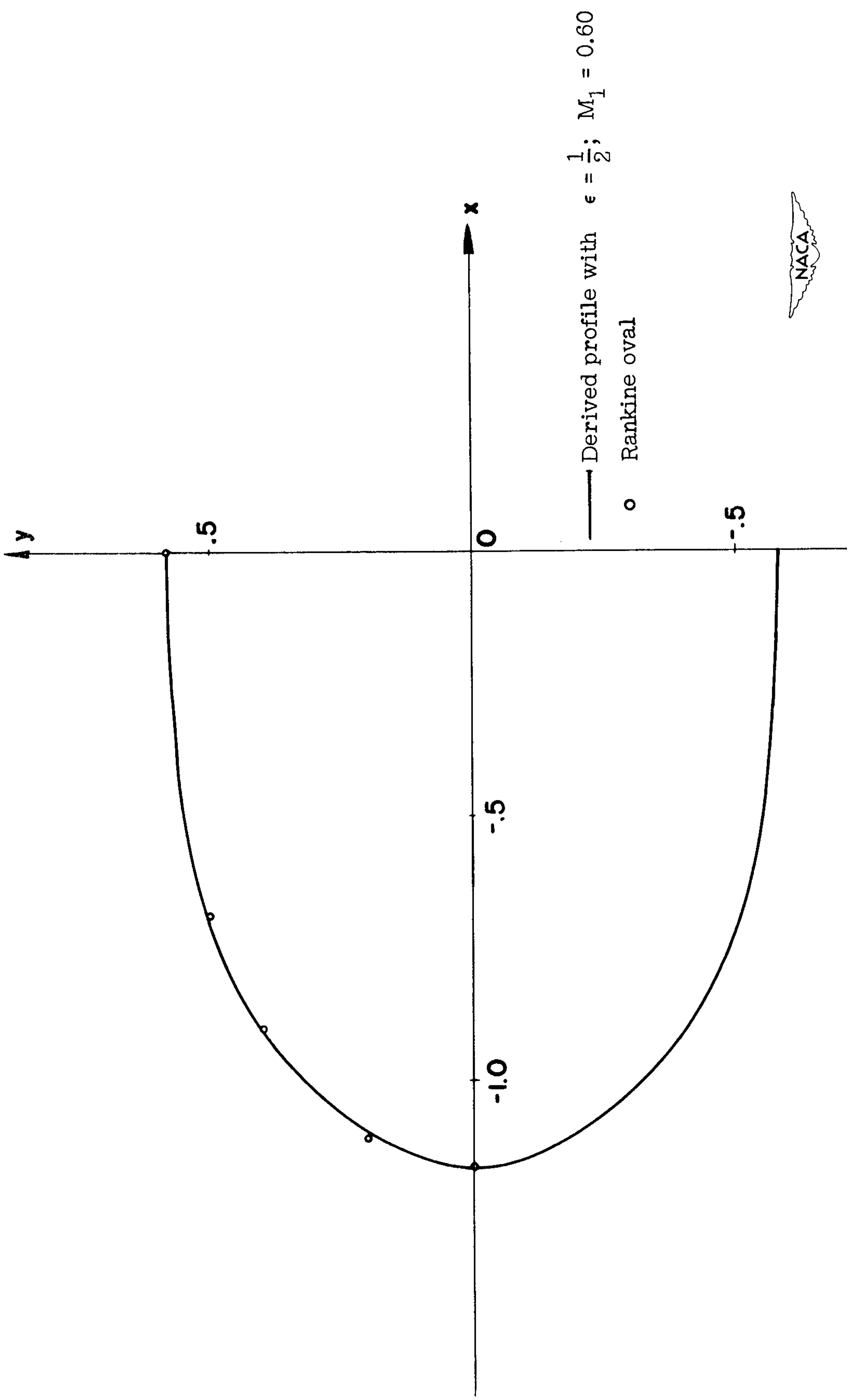

Figure 3.- Comparison of derived profile for $\epsilon = \frac{1}{2}$ and $M_1 = 0.60$ with profile given by source-sink combination for $M_1 = 0$.

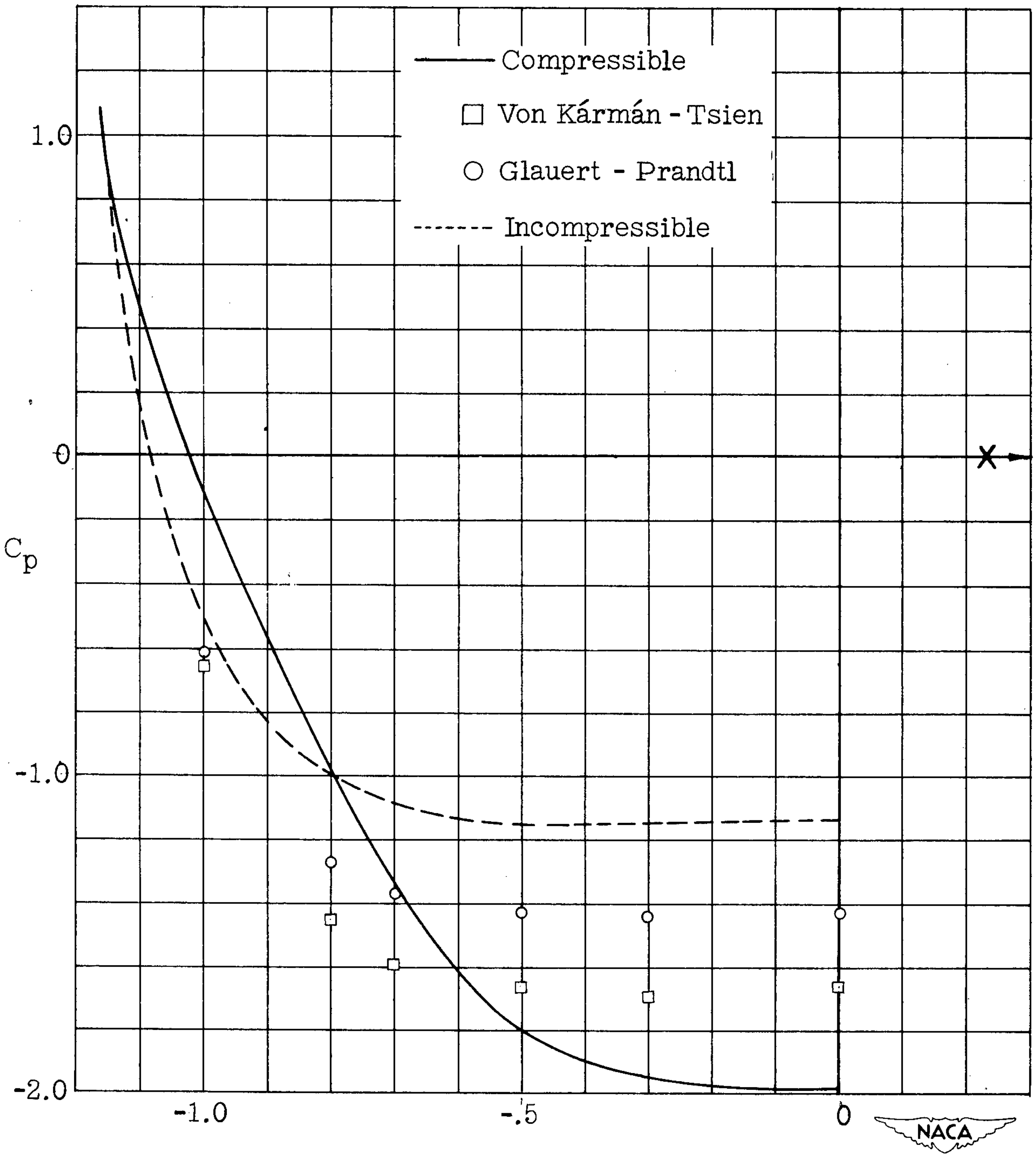

Figure 4.- Pressure distribution along major axis of ellipse.

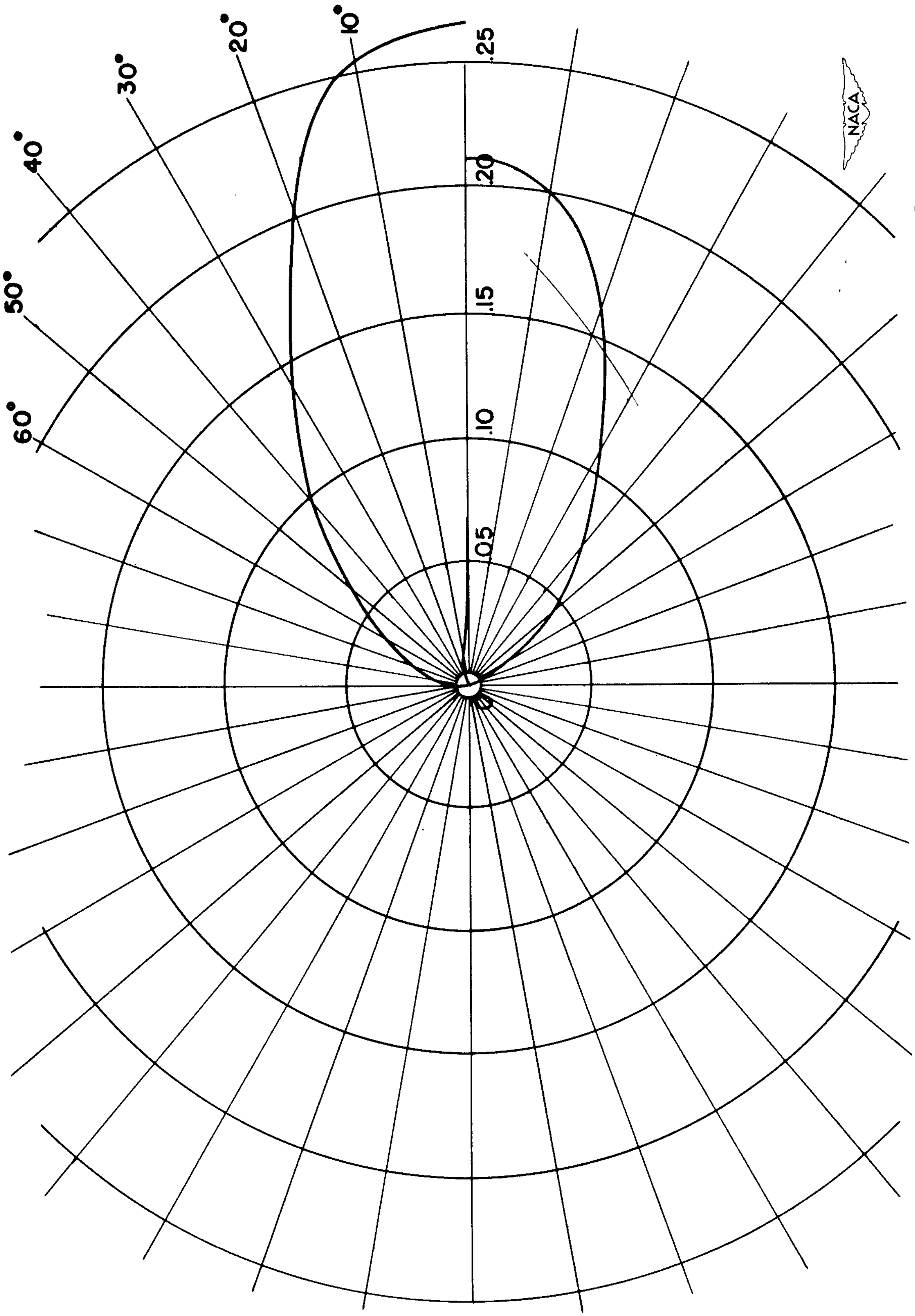

Figure 5.- Zero streamline of a compressible flow in hodograph plane. $\epsilon = \frac{1}{2}$; $M_1 = 0.60$; $\frac{\Gamma_0}{4\pi} = 0.05$.

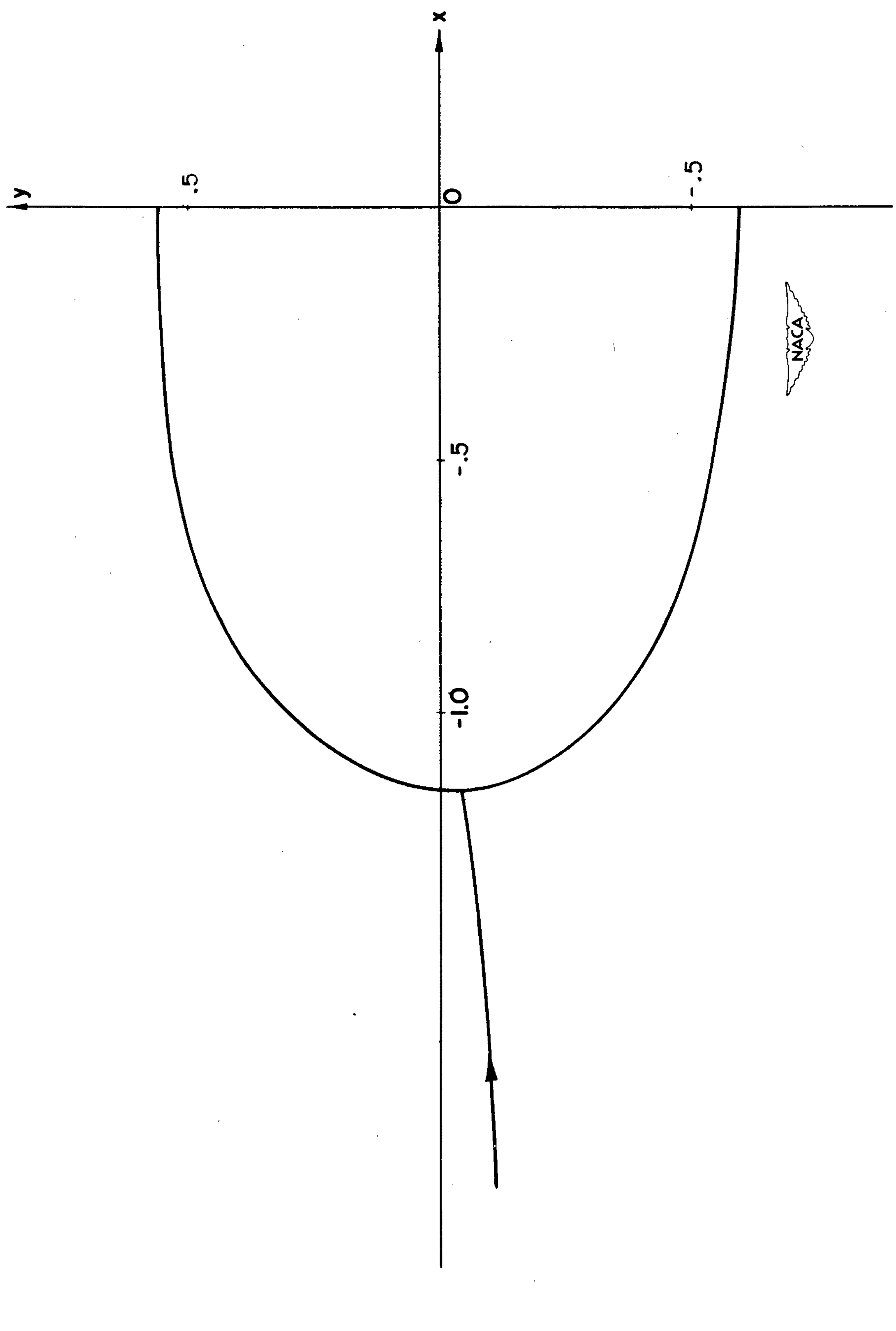

Figure 6.- Derived profile. $\epsilon = \frac{1}{2}$; $M_1 = 0.60$; $\frac{\Gamma_o}{4\pi} = 0.05$.